The Robust Price of Anarchy of Altruistic Games

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Abstract

We study the inefficiency of equilibria for various classes of games when players are (partially) altruistic. We model altruistic behavior by assuming that player i's perceived cost is a convex combination of $1 - \alpha_i$ times his direct cost and α_i times the social cost. Tuning the parameters α_i allows smooth interpolation between purely selfish and purely altruistic behavior. Within this framework, we study altruistic extensions of linear congestion games, fair cost-sharing games and valid utility games.

We derive (tight) bounds on the price of anarchy of these games for several solution concepts. Thereto, we suitably adapt the *smoothness* notion introduced by Roughgarden and show that it captures the essential properties to determine the *robust price of anarchy* of these games. Our bounds show that for congestion games and cost-sharing games, the worst-case robust price of anarchy increases with increasing altruism, while for valid utility games, it remains constant and is not affected by altruism. However, the increase in the price of anarchy is not a universal phenomenon: for symmetric singleton linear congestion games, we derive a bound on the pure price of anarchy that decreases as the level of altruism increases. Since the bound is also strictly lower than the robust price of anarchy, it exhibits a natural example in which Nash equilibria are more efficient than more permissive notions of equilibrium.

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1 Introduction

Many large-scale decentralized systems, such as infrastructure investments or traffic on roads or computer networks, bring together large numbers of individuals with different and oftentimes competing objectives. When these individuals choose actions to benefit themselves, the result is frequently suboptimal for society as a whole. This basic insight has led to a study of such systems from the viewpoint of game theory, focusing on the inefficiency of stable outcomes. Traditionally, "stable outcomes" have been associated with pure Nash equilibria of the corresponding game. The notions of *price of anarchy* [22] and *price of stability* [2] provide natural measures of the system degradation, by capturing the degradation of the worst and best Nash equilibria, respectively, compared to the socially optimal outcome.

However, the predictive power of such bounds has been questioned on (at least) two grounds:

1. The adoption of Nash equilibria as a prescriptive solution concept implicitly assumes that players are able to reach such equilibria. In particular in light of several known hardness results for finding Nash equilibria, this assumption is very suspect for computationally bounded players. In response, recent work has begun analyzing the outcomes of natural response dynamics [7, 8, 33], as well as more permissive solution concepts such as correlated or coarse correlated equilibria [3, 18, 34]. This general direction of inquiry has become known as "robust price of anarchy".
2. The assumption that players seek only to maximize their own utility is at odds with altruistic behavior routinely observed in the real world. While modeling human incentives and behavior accurately is a formidable task, several papers have proposed natural models of altruism [23, 24] and analyzed its impact on the outcomes of games [11, 12, 13, 15].

The goal of this paper is to begin a thorough investigation of the effects of relaxing both of the standard assumptions simultaneously, i.e., considering the combination of weaker solution concepts and notions of partially altruistic behavior by players. In Section 2, we formally define the *altruistic extension* of an n-player game in the spirit of past work on altruism (see [23, p. 154] and [20, 12]): player i has an associated altruism parameter α_i, and player i's cost (or payoff) is a convex combination of $(1-\alpha_i)$ times his direct cost (or payoff) and α_i times the social cost (or social welfare). By tuning the parameters α_i, this model allows smooth interpolation between pure selfishness ($\alpha_i = 0$) and pure altruism ($\alpha_i = 1$).

In order to analyze the degradation of system performance in light of partially altruistic behavior, we extend the notion of *robust price of anarchy* [33] to altruistic extensions, and show that a suitably adapted notion of *smoothness* [33] captures the properties of a system that determine its robust price of anarchy.

We use this framework to analyze three classes of games:

1. In a *cost-sharing game* [2], players choose subsets of resources, and all players choosing the same resource share its cost evenly. Thus, cost-sharing games model scenarios in which individual players have an interest in building infrastructure, and can share the cost of infrastructure that benefits several of the players. Using our framework, we derive a bound of $n/(1-\hat{\alpha})$ on the robust price of anarchy of these games, where $\hat{\alpha}$ is the maximum altruism level of a player. This bound is tight for uniformly altruistic players.
2. In *utility games* [35], players choose subsets of resources and derive utility of the chosen set. The total welfare is determined by a submodular function of the union of all chosen sets. Utility games thus model scenarios in which different players build infrastructure with different objectives, and the lack of coordination may be societally suboptimal. We derive a bound of 2 on the robust price of anarchy of these games. In particular, the bound remains at 2 regardless of the (possibly different) altruism levels of the players. This bound is tight.
3. We revisit and extend the analysis of *atomic congestion games* [32], in which players choose subsets of resources whose costs increase (linearly) with the number of players using them. Thus, they are natural models of traffic on roads or in computer networks as well as scheduling on machines, where selfish choices can lead to overcongestion of resources which would be much faster if used in moderation. Caragiannis et al. [11] recently derived a tight bound of $(5+4\alpha)/(2+\alpha)$ on the pure price of anarchy

when all players have the *same* altruism level α.[1] Our framework makes it an easy observation that their proof in fact bounds the robust price of anarchy. We generalize their bound to the case when different players have different altruism levels, obtaining a bound in terms of the maximum and minimum altruism levels. This partially answers an open question from [11]. For the special case of symmetric singleton congestion games (which corresponds to selfish scheduling on machines), we extend our study of non-uniform altruism and obtain an improved bound of $(4-2\alpha)/(3-\alpha)$ on the price of anarchy when an α-fraction of the players are entirely altruistic and the remaining players are entirely selfish.

Notice that many of these bounds on the robust price of anarchy reveal a counter-intuitive trend: at best, for utility games, the bound is independent of the level of altruism, and for congestion games and cost-sharing games, it actually *increases* in the altruism level, unboundedly so for cost-sharing games. Intuitively, this phenomenon is explained by the fact that a change of strategy by player i may affect many players. An altruistic player will care more about these other players than a selfish player; hence, an altruistic player accepts more states as "stable". This suggests that the best stable solution can also be chosen from a larger set, and the price of stability should thus decrease. Our results on the price of stability lend support to this intuition: for congestion games, we derive an upper bound on the price of stability which decreases as $2/(1+\alpha)$; similarly, for cost-sharing games, we establish an upper bound which decreases as $(1-\alpha)H_n+\alpha$.

The increase in the price of anarchy is not a universal phenomenon, demonstrated by *symmetric singleton* congestion games. Caragiannis et al. [11] showed a bound of $4/(3+\alpha)$ for pure Nash equilibria with uniformly altruistic players, which decreases with the altruism level α. Our bound of $(4-2\alpha)/(3-\alpha)$ for mixtures of entirely altruistic and selfish players is also decreasing in the fraction of entirely altruistic players. We also extend an example of Lücking et al. [25] to show that symmetric singleton congestion games may have a mixed price of anarchy arbitrarily close to 2 for arbitrary altruism levels. In light of the above bounds, this establishes that pure Nash equilibria can result in strictly lower price of anarchy than weaker solution concepts.

Related Work. Much of our analysis is based on extensions of the notion of *smoothness* as proposed by Roughgarden [33] (see Section 2.2). The basic idea is to bound the sum of cost increases of individual players switching strategies by a combination of the costs of two states. Because these types of bounds capture local improvement dynamics, they bound the price of anarchy not only for Nash equilibria, but also more general solution concepts, including coarse correlated equilibria. The smoothness notion was recently refined in the *local smoothness* framework by Roughgarden and Schoppmann [34]. They require the types of bounds described above only for nearby states, thus obtaining tighter bounds, albeit only for more restrictive solution concepts and convex strategy sets. Using the local smoothness framework, they obtained optimal upper bounds for atomic splittable congestion games. Nadav and Roughgarden [28] showed that smoothness bounds apply all the way to solution concepts called "average coarse correlated equilibrium," but not beyond.

A comparison between the costs in worst-case outcomes under solution concepts of different generality was recently undertaken by Bradonjic et al. [9] under the name "price of mediation:" specifically for the case of symmetric singleton congestion games with convex latency functions, they showed that the ratio between the most expensive correlated equilibrium and the most expensive Nash equilibrium can grow exponentially in the number of players.

Hayrapetyan et al. [19] studied the impact of "collusion" in network congestion games, where players form coalitions to minimize their collective cost. These coalitions are assumed to be formed exogeneously, i.e., conceptually, each coalition is replaced by a "super-player" that acts on behalf of its members. The authors show that collusion in network congestion games can lead to Nash equilibria that are inferior to the ones of the collusion-free game (in terms of social cost). They also derive bounds on the the price of anarchy caused by collusion. Note that the cooperation within each coalition can be interpreted as a kind of "locally" altruistic behavior, i.e., each player only cares about the cost of the members of his coalition. In a sense, the setting considered in [19] can therefore be regarded as being orthogonal to the viewpoint that we adopt in this paper: in their setting, players are assumed to be entirely altruistic but locally attached to

[1] The altruism model of [11] differs from ours in a slight technicality discussed in Section 2 (Remark 1). Therefore, various bounds we cite here are stated differently in [11].

their coalitions. In contrast, in our setting, players may have different levels of altruism but locality does not play a role.

Several recent studies investigate "irrational" player behavior in games; examples include studies on malicious (or spiteful) behavior [5, 10, 13, 21] and unpredictable (or Byzantine) behavior [8, 27, 31]. The work that is most related to our work in this context is the one by Blum et al. [8]. The authors consider repeated games in which every player is assumed to minimize his own regret. They derive bounds on the inefficiency, called *total price of anarchy*, of the resulting outcomes for certain classes of games, including congestion games and valid utility games. The exhibited bounds exactly match the respective price of anarchy and even continue to hold if only some of the players minimize their regret while the others are Byzantine. The latter result is surprising in the context of valid utility games because it means that the price of total anarchy remains at 2, even if additional players are added to the game that behave arbitrarily. Our findings allow us to draw an even more dramatic conclusion. Our bounds on the robust price of anarchy also extend to the total price of anarchy of the respective repeated games (see Section 2.3). As a consequence, our result for valid utility games implies that the price of total anarchy would remain at 2, even if the "Byzantine" players were to act altruistically. That is, while the result in [8] suggests that arbitrary behavior does not harm the inefficiency of the final outcome, our result shows that altruistic behavior does not help.

If players' altruism levels are not uniform, then even the existence of pure Nash equilibria is not obvious. Hoefer and Skopalik established it for several subclasses of atomic congestion games [20]; for the generalization of arbitrary player-specific cost functions, Milchtaich [26] showed existence for singleton congestion games, and Ackermann et al. [1] for matroid congestion games, in which the strategy space of each player is the basis of a matroid on the set of resources.

Models of Altruism. Models of altruism either identical or very similar to the one in this paper have been studied in several papers. Perhaps the first published suggestion of a similar model is due to Ledyard [23], but since then, different variations of it have been studied more extensively, e.g., [11, 12, 13, 15]. The main difference is that in some of these models, linear combinations (rather than convex combinations) are considered, e.g., with the selfish term having a factor of 1. For most of these variations, a straightforward scaling of the coefficients shows equivalence with the model we consider here. The altruism model can be naturally extended to include $\alpha_i < 0$, modeling spiteful behavior (see, e.g., [13]). While the modeling extension is natural, several results in this and other papers do not continue to hold directly for negative α_i. Our model is strictly more general than some of the previous work in that the social cost function need not be the sum of all players' costs, but rather only needs to be bounded by the sum.

Besides models based on linear combinations of individual players' costs (as well as social welfare), several other approaches have been studied. Generally, altruism or other "other-regarding" social behavior has received some attention in the behavioral economics literature (e.g., [17]). Alternative models of altruism and spite have been proposed by Levine [24], Rabin [30] and Geneakoplos et al. [16]. These models are designed more with the goal of modeling the psychological processes underlying spite or altruism (and reciprocity): they involve players forming beliefs about other players. As a result, they are well-suited for experimental work, but perhaps not as directly suited for the type of analysis in this paper.

2 Preliminaries

Let $G = (N, \{\Sigma_i\}_{i \in N}, \{C_i\}_{i \in N})$ be a finite strategic game, where $N = [n]$ is the set of players, Σ_i the strategy space of player i, and $C_i : \Sigma \to \mathbb{R}$ the cost function of player i, mapping every strategy profile $s \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ to the player's direct cost. Unless stated otherwise, we assume that every player i wants to minimize his individual cost function C_i. We also call such games *cost-minimization games*. A *social cost* function $C : \Sigma \to \mathbb{R}$ maps strategies to social costs. We require that C is *sum-bounded*, that is, $C(s) \leq \sum_{i=1}^{n} C_i(s)$ for all $s \in \Sigma$. We study *altruistic extensions* of strategic games equipped with sum-bounded social cost functions. Our definition is based on one used (among others) in [12], and similar to ones given in [11, 13, 23].

Definition 1 (Altruistic extension)**.** Let $\alpha \in [0,1]^n$. The *α-altruistic extension* of G (or simply *α-altruistic game*) is defined as the strategic game $G^\alpha = (N, \{\Sigma_i\}_{i\in N}, \{C_i^\alpha\}_{i\in N})$, where for every $i \in N$ and $s \in \Sigma$,

$$C_i^\alpha(s) \quad = \quad (1-\alpha_i)C_i(s) + \alpha_i C(s).$$

Thus, the perceived cost that player i experiences is a convex combination of his direct (selfish) cost and the social cost; we call such a player *α_i-altruistic.*[2] When $\alpha_i = 0$, player i is entirely selfish; thus, $\alpha = \mathbf{0}$ recovers the original game. A player with $\alpha_i = 1$ is entirely altruistic. Given an altruism vector $\alpha \in [0,1]^n$, we let $\hat{\alpha} = \max_{i\in N} \alpha_i$ and $\check{\alpha} = \min_{i\in N} \alpha_i$ denote the maximum and minimum altruism levels, respectively. When $\alpha_i = \alpha$ (a scalar) for all i, we call such games *uniformly α-altruistic games.*

Remark 1. In a recent paper, Caragiannis et al. [11] model uniformly altruistic players by defining the perceived cost of player i as $(1-\xi)C_i(s) + \xi(C(s) - C_i(s))$, where $\xi \in [0,1]$. It is not hard to see that in the range $\xi \in [0, \frac{1}{2}]$ this definition is equivalent to ours by setting $\alpha = \xi/(1-\xi)$ or $\xi = \alpha/(1+\alpha)$.[3]

The altruistic extension of a *payoff-maximization game*, in which players seek to maximize their payoff functions $\{\Pi_i\}_{i\in N}$, with a social welfare function Π is defined analogously to Definition 1; the only difference is that every player i wants to *maximize* Π_i^α instead of minimizing C_i^α here.

2.1 Equilibrium Concepts

We study the inefficiency of equilibria in altruistic extensions of various games. The most general equilibrium concept that we will deal with is the following one.

Definition 2 (Coarse equilibrium)**.** A *coarse equilibrium* (or *coarse correlated equilibrium*) of a game G is a probability distribution σ over $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ with the following property: if s is a random variable with distribution σ, then for each player i, and all $s_i^* \in \Sigma_i$:

$$\mathbf{E}_{s\sim\sigma}\left[C_i(s)\right] \quad \leq \quad \mathbf{E}_{s_{-i}\sim\sigma_{-i}}\left[C_i(s_i^*, s_{-i})\right], \tag{1}$$

where σ_{-i} is the projection of σ on $\Sigma_{-i} = \Sigma_1 \times \cdots \times \Sigma_{i-1} \times \Sigma_{i+1} \times \cdots \times \Sigma_n$.

The set of all coarse equilibria is also known as the *Hannan Set* (see, e.g., [36]). It includes several other solution concepts, such as correlated equilibria, mixed Nash equilibria and pure Nash equilibria. We briefly review these equilibrium notions.

Informally, the difference between a coarse equilibrium and a *correlated equilibrium* is the following: in a coarse equilibrium, it is required that a player "adheres" to s when he is informed of the distribution σ from which s is drawn. In a correlated equilibrium, a player is only required to adhere to s when he is informed of the distribution σ as well as the strategy that has been drawn for him, i.e., that he will play under s.

A *mixed Nash equilibrium* is a coarse equilibrium whose distribution σ is the product of *independent* distributions $\sigma_1, \ldots, \sigma_n$ for the players. Thus, any mixed Nash equilibrium is also a correlated equilibrium. A *pure Nash equilibrium* is a strategy profile s such that for each player i, $C_i(s) \leq C_i(s_i^*, s_{-i})$ for all $s_i^* \in \Sigma_i$. A pure Nash equilibrium is a special case of a mixed Nash equilibrium where the support of σ_i has cardinality 1 for all i.

We use PNE(G), MNE(G), CE(G), and CCE(G), to denote the set of pure Nash equilibria, mixed Nash equilibria, correlated equilibria, and coarse equilibria of a game G, respectively.

The *price of anarchy* [22] and *price of stability* [2] are natural ways of quantifying the inefficiency of equilibria for classes of games:

[2] We note that the altruistic part of an individual's perceived cost does not recursively take other players' *perceived* cost into account. Such recursive definitions of altruistic utility have been studied, e.g., by Bergstrom [6], and can be reduced to our definition under suitable technical conditions.

[3] The model of [11] with $\xi \in (\frac{1}{2}, 1]$ has players assign strictly more weight to others than to themselves, a possibility not present in our model since we consider altruism to be caring about others' costs at most as much as about one's own cost.

Definition 3 (Price of anarchy, price of stability)**.** Let $S \subseteq \Sigma$ be a set of strategy profiles for a cost-minimization game G with social cost function C, and let s^* be a strategy profile that minimizes C. We define

$$\mathrm{PoA}(S,G) = \sup\left\{\frac{C(s)}{C(s^*)} : s \in S\right\} \quad \text{and} \quad \mathrm{PoS}(S,G) = \inf\left\{\frac{C(s)}{C(s^*)} : s \in S\right\}.$$

The *coarse* (respectively *correlated, mixed, pure*) *price of anarchy* of a class of games $\mathcal{G}$ is defined as

$$\sup\{\mathrm{PoA}(S_G, G) : G \in \mathcal{G}\}, \tag{2}$$

where $S_G = \mathrm{CCE}(G)$ (respectively $\mathrm{CE}(G)$, $\mathrm{MNE}(G)$, $\mathrm{PNE}(G)$). The *coarse* (respectively *correlated, mixed, pure*) *price of stability* of a class of games is defined analogously, i.e., by replacing PoA by PoS in (2).

Notice that the price of anarchy and price of stability are defined with respect to the *original* social cost function C, not accounting for the altruistic components. This reflects our desire to understand the overall performance of the system (or strategic game), which is not affected by different *perceptions* of costs by individuals. Note, however, that if all players have a uniform altruism level $\alpha_i = \alpha \in [0,1]$ and the social cost function C is equal to the sum of all players' individual costs, then for every strategy profile $s \in \Sigma$, $C^\alpha(s) = (1 - \alpha + \alpha n)C(s)$, where $C^\alpha(s) = \sum_{i \in N} C_i^\alpha(s)$ denotes the sum of all players' perceived costs. In particular, bounding the price of anarchy with respect to C is equivalent to bounding the price of anarchy with respect to total perceived cost C^α in this case.

We extend Definition 3 in the obvious way to payoff-maximization games G with social welfare function Π by considering the ratio $\Pi(s^*)/\Pi(s)$, where s^* refers to a strategy profile maximizing Π.

2.2 Smoothness

Many proofs bounding the price of anarchy for specific games (e.g., [32, 35]) use the fact that deviating from an equilibrium to the strategy at optimum is not beneficial for any player. The addition of these inequalities, combined with suitable properties of the social cost function, then gives a bound on the equilibrium's cost. Roughgarden [33] recently captured the essence of this type of argument with his definition of (λ, μ)-*smoothness* of a game, thus providing a generic template for proving bounds on the price of anarchy. Indeed, because such arguments only reason about local moves by players, they immediately imply bounds not only for Nash equilibria, but all classes of equilibria defined in Section 2.1, as well as the outcomes of no-regret sequences of play [8, 7]. Recent work has explored both the limits of this concept [28] and a refinement requiring smoothness only in local neighborhoods [34]. The latter permits more fine-grained analysis of games, but applies only to correlated equilibria and their subclasses.

In extending the definition of smoothness to altruistic games, we have to exercise some care. Simply applying Roughgarden's definition to the new game does not work, as the social cost function we wish to bound is the sum of all direct costs *without consideration of the altruistic component.* Thus, with respect to the social cost, altruistic games are in general not sum-bounded. For this reason, we propose a slightly revised definition of (λ, μ, α)-*smoothness*; the bulk of this paper is devoted to showing that with the definition, many useful properties of Roughgarden's definition are preserved.

For notational convenience, we define $C_{-i}(s) = C(s) - C_i(s) \leq \sum_{j \neq i} C_j(s)$. Note that when the social cost is the sum of all players' costs, the inequality is an equality.

Definition 4 ((λ, μ, α)-smoothness)**.** Let G^α be a α-altruistic extension of a game with sum-bounded social cost function C. G^α is (λ, μ, α)-*smooth* iff for any two strategy profiles $s, s^* \in \Sigma$,

$$\sum_{i=1}^{n} C_i(s_i^*, s_{-i}) + \alpha_i(C_{-i}(s_i^*, s_{-i}) - C_{-i}(s)) \leq \lambda C(s^*) + \mu C(s). \tag{3}$$

For $\alpha = \mathbf{0}$, this definition coincides with Roughgarden's notion of (λ, μ)-smoothness. To gain some intuition, consider two strategy profiles $s, s^* \in \Sigma$, and a player $i \in N$ who switches from his strategy s_i under s to s_i^*, while the strategies of the other players remain fixed at s_{-i}. The contribution of player i to the left-hand side of (3) then accounts for the individual cost that player i perceives after the switch plus α_i times the difference in social cost caused by this switch exluding player i. The sum of these contributions needs to be bounded by $\lambda C(s^*) + \mu C(s)$. We will see that this definition of (λ, μ, α)-smoothness allows us to quantify the price of anarchy of some large classes of altruistic games with respect to the very broad class of coarse correlated equilibria.

2.3 Preliminary Results

We first show that many of the results in [33] following from (λ, μ)-smoothness carry over to our altruistic setting using the extended (λ, μ, α)-smoothness notion (Definition 4). Even though some care has to be taken in extending these results, most of the proofs of the propositions in this section follow along similar lines as their analogues in [33].

Proposition 1. *Let G^α be a α-altruistic game. If G^α is (λ, μ, α)-smooth with $\mu < 1$, then the coarse (and thus correlated, mixed, and pure) price of anarchy of G^α is at most $\frac{\lambda}{1-\mu}$.*

Proof. Let σ be a coarse equilibrium of G^α, s a random variable with distribution σ, and $s^* \in \Sigma$ an arbitrary strategy profile. The coarse equilibrium condition implies that for every player $i \in N$:

$$\mathbf{E}\left[(1-\alpha_i)C_i(s) + \alpha_i C(s)\right] \leq \mathbf{E}\left[(1-\alpha_i)C_i(s_i^*, s_{-i}) + \alpha_i C(s_i^*, s_{-i})\right].$$

By linearity of expectation, for every player $i \in N$:

$$\mathbf{E}\left[C_i(s)\right] \leq \mathbf{E}\left[C_i(s_i^*, s_{-i}) + \alpha_i(C(s_i^*, s_{-i}) - C_i(s_i^*, s_{-i})) - \alpha_i(C(s) - C_i(s))\right].$$

By summing over all players and using linearity of expectation, we obtain

$$\mathbf{E}\left[C(s)\right] \leq \mathbf{E}\left[\sum_{i=1}^{n} C_i(s_i^*, s_{-i}) + \alpha_i(C_{-i}(s_i^*, s_{-i}) - C_{-i}(s))\right].$$

Now we use the smoothness property (3) to conclude

$$\mathbf{E}\left[C(s)\right] \leq \mathbf{E}\left[\lambda C(s^*) + \mu C(s)\right] = \lambda C(s^*) + \mu \mathbf{E}\left[C(s)\right].$$

Solving for $\mathbf{E}\left[C(s)\right]$ now proves the claim. As coarse equilibria include correlated equilibria, mixed Nash equilibria and pure Nash equilibria, the correlated, mixed, and pure price of anarchy are thus also bounded by $\frac{\lambda}{1-\mu}$. □

As we show later, for many important classes of games, the bounds obtained by (λ, μ, α)-smoothness arguments are actually tight, even for pure Nash equilibria. Therefore, as in [33], we define the *robust price of anarchy* as the best possible bound on the coarse price of anarchy obtainable by a (λ, μ, α)-smoothness argument.

Definition 5. The *robust price of anarchy* of a α-altruistic game G^α is defined as

$$\mathrm{RPoA}_G(\alpha) = \inf\left\{\tfrac{\lambda}{1-\mu} : G^\alpha \text{ is } (\lambda, \mu, \alpha)\text{-smooth, } \mu < 1\right\}.$$

For a class $\mathcal{G}$ of games, we define $\mathrm{RPoA}_{\mathcal{G}}(\alpha) = \sup\{\mathrm{RPoA}_G(\alpha) : G \in \mathcal{G}\}$. We omit the subscript when the game (or class of games) is clear from the context.

The smoothness condition also proves useful in the context of no-regret sequences and the *price of total anarchy*, introduced by Blum et al. [8].

Proposition 2. *Let s^* be a strategy profile minimizing the social cost function C of an α-altruistic game G^α, and $s^1, \dots, s^T$ a sequence of strategy profiles in which every player $i \in N$ experiences vanishing average external regret, i.e.,*

$$\sum_{t=1}^{T} C_i^\alpha(s^t) \leq \left(\min_{s_i' \in \Sigma_i} \sum_{t=1}^{T} C_i^\alpha(s_i', s_{-i}^t) \right) + o(T).$$

The average cost of this sequence of T strategy profiles then satisfies

$$\frac{1}{T} \sum_{t=1}^{T} C(s^t) \leq RPoA(\alpha) \cdot C(s^*) \quad \text{as } T \to \infty.$$

Proof. Consider a sequence $s^1, \dots, s^T$ of strategy profiles of an α-altruistic game G^α that is (λ, μ, α)-smooth with $\mu < 1$. For every $i \in N$ and $t \in \{1, \dots, T\}$, define

$$\delta_i^\alpha(s^t) = C_i^\alpha(s^t) - C_i^\alpha(s_i^*, s_{-i}^t).$$

Let $\Delta(s^t) = \sum_{i=1}^{n} \delta_i^\alpha(s^t)$. We have

$$\begin{aligned} \Delta(s^t) &= \sum_{i=1}^{n} C_i^\alpha(s^t) - C_i^\alpha(s_i^*, s_{-i}^t) \\ &= \sum_{i=1}^{n} \big((1-\alpha_i) C_i(s^t) + \alpha_i C(s^t) - \big((1-\alpha_i) C_i(s_i^*, s_{-i}^t) + \alpha_i C(s_i^*, s_{-i}^t)\big)\big) \\ &= C(s^t) - \sum_{i=1}^{n} \big(C_i(s_i^*, s_{-i}^t) + \alpha_i (C_{-i}(s_i^*, s_{-i}^t) - C_{-i}(s^t))\big). \end{aligned}$$

Exploiting the (λ, μ, α)-smoothness property, we obtain

$$C(s^t) \leq \tfrac{\lambda}{1-\mu} C(s^*) + \tfrac{1}{1-\mu} \Delta(s^t). \tag{4}$$

Suppose that $s^1, \dots, s^T$ is a sequence of strategy profiles in which every player experiences vanishing average external regret, i.e.,

$$\sum_{t=1}^{T} C_i^\alpha(s^t) \leq \left(\min_{s_i' \in \Sigma_i} \sum_{t=1}^{T} C_i^\alpha(s_i', s_{-i}^t) \right) + o(T).$$

We obtain that for every player $i \in N$:

$$\frac{1}{T} \sum_{t=1}^{T} \delta_i^\alpha(s^t) \leq \frac{1}{T} \left(\sum_{t=1}^{T} C_i^\alpha(s^t) - \min_{s_i' \in \Sigma_i} \sum_{t=1}^{T} C_i^\alpha(s_i', s_{-i}^t) \right) = o(1).$$

Using this inequality and (4), we obtain that the average cost of the sequence of T strategy profiles is

$$\frac{1}{T} \sum_{t=1}^{T} C(s^t) \leq \frac{\lambda}{1-\mu} C(s^*) + \frac{1}{1-\mu} \sum_{i=1}^{n} \left(\frac{1}{T} \sum_{t=1}^{T} \delta_i^\alpha(s^t) \right) \overset{T \to \infty}{\longrightarrow} \frac{\lambda}{1-\mu} C(s^*).$$

□

Roughgarden [33, Proposition 2.6] shows that for games that have an *underestimating exact potential function*, best response dynamics[4] converge rapidly to a strategy profile of social cost close to the robust

[4]Best response dynamics are a natural way of searching for a pure Nash equilibrium: if the current strategy profile is not a Nash equilibrium, then pick a player who can improve his cost and change his strategy to one that minimizes his cost.

price of anarchy times the optimum social cost of the game; see [33] for a precise statement of this result and the accompanying definitions. Proposition 2.6 in [33] and its proof straightforwardly carry over to (λ, μ, α)-smooth games that have such an underestimating exact potential function.

The results in this section continue to hold for altruistic extensions of payoff-maximization games if we adapt Definition 4 as follows. Let G^α be an α-altruistic extension of a payoff-maximization game with social welfare function Π. Define $\Pi_{-i}(s) = \Pi(s) - \Pi_i(s)$. G^α is (λ, μ, α)-smooth iff for every two strategy profiles $s, s^* \in \Sigma$,

$$\sum_{i=1}^{n}(\Pi_i(s_i^*, s_{-i}) + \alpha_i(\Pi_{-i}(s_i^*, s_{-i}) - \Pi_{-i}(s))) \geq \lambda\Pi(s^*) - \mu\Pi(s). \tag{5}$$

Given this smoothness definition, all the results above hold when we replace $\frac{\lambda}{1-\mu}$ by $\frac{1+\mu}{\lambda}$ and $\mu < 1$ by $\mu > -1$ in Definition 5.

3 Fair Cost-sharing Games

A *fair cost-sharing game* $G = (N, E, \{\Sigma_i\}_{i\in N}, \{c_e\}_{e\in E})$ is characterized by a set E of *resources* (or *facilities*), and strategy sets $\Sigma_i \subseteq 2^E$; that is, players' strategies $s_i \in \Sigma_i$ are subsets of resources. Given a strategy profile $s \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_n$, we define $x_e(s) = |\{i \in N \ : \ e \in s_i\}|$ as the number of players that use resource $e \in E$ under s. Let $U(s)$ be the set of resources that are used under s, i.e., $U(s) = \bigcup_{i\in N} s_i$. Each facility $e \in E$ has a cost c_e which is evenly shared by all players using e, i.e., the direct cost of player i is $C_i(s) = \sum_{e\in s_i} c_e/x_e(s)$. The social cost function is $C(s) = \sum_{i=1}^{n} C_i(s) = \sum_{e\in U(s)} c_e$.

It is well-known that the pure price of anarchy of fair cost-sharing games is n [29]. We show that it can get significantly worse in the presence of altruistic players: the following theorem gives a much worse upper bound, which we subsequently show to be tight.

Theorem 3. *The robust price of anarchy of α-altruistic cost-sharing games is at most $\frac{n}{1-\hat{\alpha}}$ (with $n/0 = \infty$).*

Proof. The claim is true for $\hat{\alpha} = 1$ because $\mathrm{RPoA}(\alpha) \leq \infty$ holds trivially. We show that G^α is $(n, \hat{\alpha}, \alpha)$-smooth for $\hat{\alpha} \in [0, 1)$. Let s and s^* be two strategy profiles. Fix an arbitrary player $i \in N$. We have

$$C(s_i^*, s_{-i}) - C(s) = \sum_{e\in U(s_i^*, s_{-i})} c_e - \sum_{e\in U(s)} c_e \leq \sum_{e\in s_i^* \setminus U(s)} c_e.$$

We use this inequality to obtain the following bound:

$$\begin{aligned}(1-\alpha_i)C_i(s_i^*, s_{-i}) + \alpha_i(C(s_i^*, s_{-i}) - C(s)) &\leq (1-\alpha_i)\sum_{e\in s_i^*} \frac{c_e}{x_e(s_i^*, s_{-i})} + \alpha_i \sum_{e\in s_i^*\setminus U(s)} \frac{c_e}{x_e(s_i^*, s_{-i})} \\ &\leq \sum_{e\in s_i^*} \frac{c_e}{x_e(s_i^*, s_{-i})} \leq \sum_{e\in s_i^*} \frac{n \cdot c_e}{x_e(s^*)}.\end{aligned}$$

The first inequality holds because $x_e(s_i^*, s_{-i}) = 1$ for every $e \in s_i^* \setminus U(s)$, and the last inequality follows from $x_e(s_i^*, s_{-i}) \geq x_e(s^*)/n$ for every $e \in s_i^*$. The left-hand side of the smoothness condition (3) is equivalent to

$$\begin{aligned}\sum_{i=1}^{n}((1-\alpha_i)C_i(s_i^*, s_{-i}) + \alpha_i(C(s_i^*, s_{-i}) - C(s)) + \alpha_i C_i(s)) &\leq \sum_{i=1}^{n}\left(\sum_{e\in s_i^*} \frac{n \cdot c_e}{x_e(s^*)}\right) + \hat{\alpha}C(s) \\ &= nC(s^*) + \hat{\alpha}C(s).\end{aligned}$$

We conclude that the robust price of anarchy is at most $\frac{n}{1-\hat{\alpha}}$. Example 1 shows that this bound is tight, even for pure Nash equilibria. $\square$

Example 1. Consider the cost-sharing game in which n players can choose between two different facilities e_1 and e_2 of cost 1 and $n/(1-\alpha)$, respectively. Let $s^* = (e_1, \dots, e_1)$ and $s = (e_2, \dots, e_2)$ refer to the strategy profiles in which every player chooses e_1 and e_2, respectively. Then $C(s^*) = 1$ and $C(s) = n/(1-\alpha)$. Note that s is a pure Nash equilibrium of the α-altruistic extension of this game because for every player i we have

$$(1-\alpha)C_i(s) + \alpha C(s) = 1 + \alpha \frac{n}{1-\alpha} = C_i^\alpha(\{e_1\}, s_{-i}).$$

The pure price of anarchy is therefore at least $n/(1-\alpha)$.

We turn to the pure price of stability of uniformly α-altruistic cost-sharing games. Clearly, an upper bound on the pure price of stability extends to the mixed, correlated and coarse price of stability. As opposed to the price of anarchy, the price of stability does improve with increased altruism. The proof of the following proposition exploits a standard technique to bound the pure price of stability of exact potential games (see, e.g., [29]).

Proposition 4. *The pure price of stability of uniformly α-altruistic cost-sharing games is at most* $(1-\alpha)H_n + \alpha$.

Proof. Let G^α be a uniformly α-altruistic cost-sharing game. It is not hard to verify that G^α is an exact potential game with potential function $\Phi^\alpha(s) = (1-\alpha)\Phi(s) + \alpha C(s)$, where $\Phi(s) = \sum_{e \in E} \sum_{i=1}^{x_e(s)} c_e/i$. Observe that

$$\Phi^\alpha(s) = (1-\alpha) \sum_{e \in E} \sum_{i=1}^{x_e(s)} \frac{c_e}{i} + \alpha \sum_{e \in U(s)} c_e \leq ((1-\alpha)H_n + \alpha) \sum_{e \in U(s)} c_e = ((1-\alpha)H_n + \alpha)C(s).$$

We therefore have that $C(s) \leq \Phi^\alpha(s) \leq ((1-\alpha)H_n + \alpha)C(s)$.

Let s be a strategy profile that minimizes Φ^α, and let s^* be an optimal strategy profile that minimizes the social cost function C. Note that s is a pure Nash equilibrium of G^α. We have

$$C(s) \leq \Phi^\alpha(s) \leq \Phi^\alpha(s^*) \leq ((1-\alpha)H_n + \alpha)C(s^*),$$

which proves the claim. □

4 Valid Utility Games

A valid utility game [35] is a payoff maximization game given by $G = (N, E, \{\Sigma_i\}_{i \in N}, \{\Pi_i\}_{i \in N}, V)$, where E is a ground set of resources, the strategy sets Σ_i are subsets of E, Π_i is the payoff function of player i, and V is a submodular[5] and non-negative function on E. Every player strives to maximize his individual payoff function Π_i.

For a strategy profile $s \in \Sigma$, let $U(s) = \bigcup_{i \in N} s_i \subseteq E$ be the union of all players' strategies under s. The social welfare function $\Pi : \Sigma \to \mathbb{R}$ to be maximized is $\Pi(s) = V(U(s))$, and thus depends only on the union of the players' chosen strategies, evaluated by V. The individual payoff functions of all players $i \in N$ are assumed to satisfy[6] $\Pi_i(s) \geq \Pi(s) - \Pi(\emptyset, s_{-i})$ for every strategy profile $s \in \Sigma$. Intuitively, this means that the individual payoff of a player is at least his contribution to the social welfare. Moreover, it is assumed that $\Pi(s) \geq \sum_{i=1}^n \Pi_i(s)$ for every $s \in \Sigma$. See [35] for a detailed description and justification of these assumptions.

Examples of games falling into this framework include natural game-theoretic variants of the facility location, k-median and network routing problems [35]. Vetta [35] proved a bound of 2 on the pure price of anarchy for valid utility games with non-decreasing V, and Roughgarden showed in [33] how this bound is achieved via a (λ, μ)-smoothness argument. We extend this result to altruistic extensions of these games.

[5]For a finite set E, a function $f : 2^E \to \mathbb{R}$ is *submodular* iff $f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$ for any $A \subseteq B \subseteq E, x \in E$.
[6]We abuse notation and write $\Pi(\emptyset, s_{-i})$ to denote $V(U(s) \backslash s_i)$.

Theorem 5. *The robust price of anarchy of α-altruistic valid utility games is* 2.

Note that in the statement above, α is a vector, i.e., the claim holds for arbitrary non-uniform altruism.

Proof. We show that the α-altruistic extension G^α of a valid utility game is $(1, 1, \alpha)$-smooth.

Fix two strategy profiles $s, s^* \in \Sigma$ and consider an arbitrary player $i \in N$. By assumption, we have $\Pi_i(s) \geq \Pi(s) - \Pi(\emptyset, s_{-i})$. Therefore, for each player $i \in N$,

$$\begin{aligned} \Pi(s_i^*, s_{-i}) - \Pi(s) + \Pi_i(s) &= (\Pi(s_i^*, s_{-i}) - \Pi(\emptyset, s_{-i})) - (\Pi(s) - \Pi(\emptyset, s_{-i})) + \Pi_i(s) \\ &\geq \Pi(s_i^*, s_{-i}) - \Pi(\emptyset, s_{-i}). \end{aligned} \tag{6}$$

Now let $U_i = \bigcup_{j=1}^n s_j \cup \bigcup_{j=1}^i s_j^*$. Summing over all $i \in N$,

$$\begin{aligned} \sum_{i=1}^n((1-\alpha_i)\Pi_i(s_i^*, s_{-i}) + \alpha_i(\Pi(s_i^*, s_{-i}) - \Pi(s) + \Pi_i(s))) &\geq \sum_{i=1}^n(\Pi(s_i^*, s_{-i}) - \Pi(\emptyset, s_{-i})) \\ &= \sum_{i=1}^n(V(U(s_i^*, s_{-i})) - V(U(s) \setminus s_i)) \\ &\geq \sum_{i=1}^n(V(U_i) - V(U_{i-1})) \\ &\geq \Pi(s^*) - \Pi(s). \end{aligned}$$

Here, the first inequality follows from (6) and because $\Pi_i(s) \geq \Pi(s) - \Pi(\emptyset, s_{-i})$ for every i, the second inequality holds because V is submodular, and the final inequality follows from V being non-decreasing. We conclude that G^α is $(1, 1, \alpha)$-smooth, which proves an upper bound of 2 on the robust price of anarchy. This bound is tight, as shown by Example 2. □

Example 2. Consider a valid utility game G with two players $N = \{1, 2\}$, a ground set $E = \{1, 2\}$ of two elements and strategy sets $\Sigma_1 = \{\{1\}, \{2\}\}$, $\Sigma_2 = \{\emptyset, \{1\}\}$. Define $V(S) = |S|$ for every subset $S \subseteq E$. Note that V is non-negative, non-decreasing and submodular.

For a given strategy profile $s \in \Sigma$, the individual profits $\Pi_1(s)$ and $\Pi_2(s)$ of player 1 and player 2, respectively, are defined as follows: $\Pi_1(s) = 1$ for all strategy profiles s. $\Pi_2(s) = 1$ if $s = (\{2\}, \{1\})$ and $\Pi_2(s) = 0$ otherwise. It is not hard to verify that for every player i and every strategy profile $s \in \Sigma$ we have $\Pi_i(s) \geq \Pi(s) - \Pi(\emptyset, s_{-i})$. Moreover, $\Pi(s) \geq \Pi_1(s) + \Pi_2(s)$ for every $s \in \Sigma$. We conclude that G is a valid utility game.

Let $\alpha \in [0, 1]^2$, and consider the α-altruistic extension G^α of G. We claim that $s = (\{1\}, \emptyset)$ is a pure Nash equilibrium of G^α: the profit of player 1 under s is $(1-\alpha_1)+\alpha_1 = 1$. His profit remains 1 if he switches to strategy $\{2\}$. The profit of player 2 under s is α_2. If he switches to strategy $\{1\}$, then his profit is α_2 as well. Thus, s is a pure Nash equilibrium. Since $\Pi(s) = 1$ and $\Pi((\{2\}, \{1\})) = 2$, the pure price of anarchy of G is 2.

5 Congestion Games

In an *atomic congestion game* $G = (N, E, \{\Sigma_i\}_{i \in N}, \{d_e\}_{e \in E})$, players' strategies are again subsets of facilities, $\Sigma_i \subseteq 2^E$. Each facility $e \in E$ has an associated *delay function* $d_e : \mathbb{N} \to \mathbb{R}$. As in Section 3, we write $x_e(s)$ for the number of players using facility e. Player i's cost is $C_i(s) = \sum_{e \in s_i} d_e(x_e(s))$, and the social cost is $C(s) = \sum_{i=1}^n C_i(s)$. We focus on *linear* congestion games, i.e., the delay functions are of the form $d_e(x) = a_e x + b_e$, where a_e, b_e are non-negative rational numbers. Pure Nash equilibria of altruistic extensions of linear congestion games always exist [20]; this may not be the case for arbitrary (non-linear) congestion games.

The PoA of linear congestion games is known to be $\frac{5}{2}$ [14]. Recently, Caragiannis et al. [11] extended this result to linear congestion games with uniformly altruistic players. Applying the transformation outlined in Remark 1, their result can be stated as follows:

Theorem 6 (Caragiannis et al. [11])**.** *The pure price of anarchy of uniformly α-altruistic linear congestion games is at most $\frac{5+4\alpha}{2+\alpha}$.*

The proof in [11] implicitly uses a smoothness argument in the framework we define here for altruistic games. Thus, without any additional work, our framework allows the extension of Theorem 6 to the robust PoA. Caragiannis et al. [11] also showed that the bound of Theorem 6 is asymptotically tight. A simpler example (given below) proves tightness of this bound (not only asymptotically). Thus, the robust price of anarchy is exactly $\frac{5+4\alpha}{2+\alpha}$. We give a refinement of Theorem 6 to non-uniform altruism distributions, obtaining a bound in terms of the maximum and minimum altruism levels.

Theorem 7. *The robust price of anarchy of α-altruistic linear congestion games is at most $\frac{5+2\hat{\alpha}+2\check{\alpha}}{2-\hat{\alpha}+2\check{\alpha}}$.*

As a first step, we show that without loss of generality, we can focus on simpler instances of linear congestion games.

Lemma 8. *Without loss of generality, all delay functions are of the form $d_e(x) = x$.*

Proof. First, we may assume that for every delay function d_e, the a_e and b_e coefficients are integers. This can be ensured by multiplying all coefficients among all facilities by their least common multiple. In the resulting game, all coefficients are integers, the price of anarchy is the same, and so is the set of all equilibria.

Next, we can assume that $b_e = 0$ for all $e \in E$. To show this, we replace any facility $e \in E$ with delay function $d(x) = a_e x + b_e$ by $n+1$ facilities $e_0, \ldots, e_n$ with delay functions $d_{e_0}(x) = a_e x$ and $d_{e_i}(x) = b_e x$ for $1 \leq i \leq n$. We then adapt the strategy space Σ_i of each player i as follows: we replace every strategy $s_i \in \Sigma_i$ in which e occurs by the strategy $s_i \setminus \{e\} \cup \{e_0, e_i\}$. There is an obvious bijection between the strategy profiles in the original game and those in the new game, preserving the values of individual cost functions and the social cost function. (Notice that this construction exploits the fact that all players have unit weight, and would not carry over to weighted congestion games.)

Finally, for the same reason, we can also assume that $a_e = 1$ for all $e \in E$. We replace e with facilities $e_1, \ldots, e_{a_e}$, each having delay function $d_{e_i}(x) = x$, and adapt the strategy space Σ_i of each player i by replacing each strategy s_i in which e occurs by $s_i \setminus \{e\} \cup \{e_1, \ldots, a_{a_e}\}$. Now, all delay functions are $d_e(x) = x$. □

The next step in the proof of Theorem 7 is the following technical lemma:

Lemma 9. *For every two non-negative integers x, y and $\hat{\alpha}, \check{\alpha} \in [0,1]$ with $\hat{\alpha} \geq \check{\alpha}$,*

$$((1+\hat{\alpha})x+1)y + \check{\alpha}(1-x)x \leq \frac{5+2\hat{\alpha}+2\check{\alpha}}{3}y^2 + \frac{1+\hat{\alpha}-2\check{\alpha}}{3}x^2.$$

To prove this lemma, we make use of the following result:

Lemma 10. *For all $x, y \in \mathbb{N}_0$, $\alpha \in [0,1]$ and $\beta \in [0,1]$, it holds that*

$$((1+\alpha)x+1)y + \beta\alpha(1-x)x \leq (2+\alpha-\gamma)y^2 + \gamma x^2$$

for all $\gamma \in [\frac{1}{3}(1+\alpha-2\beta\alpha), 1+\alpha]$.

Proof. The inequality is equivalent to

$$((1+\alpha)x+1)y + \beta\alpha(1-x)x - (2+\alpha)y^2 \leq \gamma(x^2-y^2).$$

Assume that $x = y$. The inequality is then trivially satisfied because $x \leq x^2$ for all $x \in \mathbb{N}_0$. Next suppose that $x > y$. Then

$$\gamma \geq \frac{((1+\alpha)x+1)y + \beta\alpha(1-x)x - (2+\alpha)y^2}{x^2-y^2}.$$

We show that the maximum of the expression on the right-hand side is attained by $x = 2$ and $y = 1$. First, we fill in these values and conclude that for these values, $\gamma \geq \frac{1}{3}(1 + \alpha - 2\beta\alpha) \geq 0$. We now write x as $y + a, a \geq 1$, and rewrite the right-hand side as

$$f(y, a) = \frac{(1 + \alpha)y + \beta\alpha}{2y + a} + \frac{(1 + \beta\alpha)(y - y^2)}{a(2y + a)} - \beta\alpha. \tag{7}$$

Because we know that there are choices of x and a for which $f(y, a)$ is positive (e.g., when $y = 1$ and $a = 1$), and because a only occurs in the denominators, we know that (7) reaches its maximum when $a = 1$. So we assume $a = 1$. When we then fill in $y = 0$, we see that $f(0, 1) = 0$, so $f(1, 1) \geq f(0, 1)$. When $y > 1$ we can write y as $w + 2$, where $w \geq 0$, and we can now further rewrite $f(y, a)$ as

$$f(w + 2, 1) = \frac{2\alpha - 6\beta\alpha}{2w + 5} - \frac{(2 - \alpha + 5\beta\alpha)w + (1 + \beta\alpha)w^2}{2w + 5} \leq \frac{2\alpha - 6\beta\alpha}{2w + 5}.$$

When $2\alpha - 6\beta\alpha$ is negative, this term is certainly less than $f(1, 1)$. When $2\alpha - 6\beta\alpha$ is positive, we have

$$f(w + 2, 1) \leq \frac{2\alpha - 6\beta\alpha}{2w + 5} \leq \frac{2\alpha - 6\beta\alpha}{5} \leq \frac{1}{3}(2\alpha - 6\beta\alpha) \leq \frac{1}{3}(1 + \alpha - 2\beta\alpha) = f(1, 1).$$

This shows that $\gamma \geq f(1, 1) = \frac{1}{3}(1 + \alpha - 2\beta\alpha)$.

The final case is when $x < y$. Then,

$$\gamma \leq \frac{(2 + \alpha)y^2 - ((1 + \alpha)x + 1)y - \beta\alpha(1 - x)x}{y^2 - x^2}.$$

We show that the minimum of the expression on the right-hand side is attained by $x = 0$ and $y = 1$. First, we fill in these values and conclude that for these values, $\gamma \leq 1 + \alpha$. We now write y as $x + a$, $a \geq 1$, and rewrite the right-hand side as

$$g(x, a) = \frac{(1 + \beta\alpha)x^2 - (1 + a + (a + \beta)\alpha)x - a}{a(2x + a)} + 2 + \alpha.$$

Suppose first that $x = 0$ and that $a \geq 2$. Then we can write a as $1 + b$, $b > 0$, and therefore

$$f(0, 1 + b) = 2 + \alpha - \frac{1}{1 + b} \geq \frac{3}{2} + \alpha \geq 1 + \alpha = f(0, 1).$$

When $x \geq 1$, we can write x as $1 + b$, $b \geq 0$. We then have

$$f(1 + b, a) = 2 + \alpha - \frac{2 + \alpha + (1 - \alpha)b}{2b + 2 + a} + \frac{(1 + \beta\alpha)(b^2 + b)}{a(2b + 2 + a)}.$$

The last of these terms is positive, hence

$$\begin{aligned} f(1 + b, a) &\geq 2 + \alpha - \frac{2 + \alpha + (1 - \alpha)b}{2b + 2 + a} \geq 2 + \alpha - \frac{2 + 1 + b}{2b + 2 + a} \\ &\geq 2 + \alpha - 1 = 1 + \alpha = f(0, 1). \end{aligned}$$

This shows that $\gamma \leq f(0, 1) = 1 + \alpha$. □

Now we can complete the proof of Lemma 9.

Proof of Lemma 9. Choose $\beta \in [0, 1]$ such that $\check{\alpha} = \beta\hat{\alpha}$. Using Lemma 10 above, we obtain

$$((1 + \hat{\alpha})x + 1)y + \check{\alpha}(1 - x)x = ((1 + \hat{\alpha})x + 1)y + \beta\hat{\alpha}(1 - x)x \leq (2 + \hat{\alpha} - \gamma)y^2 + \gamma x^2,$$

where $\gamma \in [\frac{1}{3}(1 + \hat{\alpha} - 2\beta\hat{\alpha}), 1 + \hat{\alpha}]$. By choosing $\gamma = \frac{1}{3}(1 + \hat{\alpha} - 2\beta\hat{\alpha})$, we obtain

$$((1 + \hat{\alpha})x + 1)y + \check{\alpha}(1 - x)x \leq \frac{5 + 2\hat{\alpha} + 2\beta\hat{\alpha}}{3}y^2 + \frac{1 + \hat{\alpha} - 2\beta\hat{\alpha}}{3}x^2.$$

Substituting $\beta\hat{\alpha} = \check{\alpha}$ yields the claim. □

We remark that the choice of γ in the proof above has been made in order to minimize the expression $\lambda/(1-\mu)$ (which is an increasing function in γ).

Lemma 9 is essentially the part that generalizes the proof in [11], and allows us to complete the proof of Theorem 7.

Proof of Theorem 7. We show that the α-altruistic extension G^α of a linear congestion game is $(\frac{1}{3}(5+2\hat{\alpha}+2\check{\alpha}), \frac{1}{3}(1+\hat{\alpha}-2\check{\alpha}), \alpha)$-smooth.

Let s and s^* be two strategy profiles, and write $x_e = x_e(s), x_e^* = x_e(s^*)$. The left-hand side of the smoothness condition (3) is equivalent to

$$\begin{aligned}
&\sum_{i=1}^{n} ((1-\alpha_i)C_i(s_i^*, s_{-i}) + \alpha_i(C(s_i^*, s_{-i}) - C(s)) + \alpha_i C_i(s)) \\
&= \sum_{i=1}^{n} \left((1-\alpha_i) \left(\sum_{e \in s_i^* \setminus s_i} (x_e + 1) + \sum_{e \in s_i \cap s_i^*} x_e \right) + \alpha_i \left(\sum_{e \in s_i^* \setminus s_i} (2x_e + 1) + \sum_{e \in s_i \setminus s_i^*} (1 - 2x_e) \right) + \alpha_i C_i(s) \right) \\
&\leq \sum_{i=1}^{n} \left(\sum_{e \in s_i^*} ((1+\alpha_i)x_e + 1) + \alpha_i \sum_{e \in s_i} (1 - x_e) \right) \\
&\leq \sum_{e \in E} (((1+\hat{\alpha})x_e + 1)x_e^* + \check{\alpha}(1-x_e)x_e).
\end{aligned}$$

In the above derivation, the first inequality follows from the fact that $(1-\alpha_i)x_e \leq (1+\alpha_i)x_e + 1 + \alpha_i(1-2x_e)$ for every $e \in s_i \cap s_i^*$. Therefore, it is possible to simply replace all the $(1-\alpha_i)x_e$ (in the third summation operator of the left hand side of the first inequality) by $(1+\alpha_i)x_e + 1 + \alpha_i(1-2x_e)$, write $C_i(s)$ as $\sum_{e \in s_i} x_e$, and finally rewrite the resulting expression into the form of the right hand side of the first inequality. The second inequality holds because for every $i \in N$ and $e \in s_i$, $1 - x_e \leq 0$ and by the definition of $\hat{\alpha}$ and $\check{\alpha}$. The bound on the robust price of anarchy now follows from Lemma 9. □

The following is a simple example that shows that the bound of $\frac{5+4\alpha}{2+\alpha}$ on the robust price of anarchy for uniformly α-altruistic linear congestion games is tight, even for pure Nash equilibria. It slightly improves the lower bound example of [11], because it is simpler and it shows tightness of the bound not only asymptotically.

Example 3. Consider a game with six resources $E = E_1 \cup E_2$, $E_1 = \{h_0, h_1, h_2\}$, $E_2 = \{g_0, g_1, g_2\}$ and three α-altruistic players. The delay functions are given by $d_e(x) = (1+\alpha)x$ for $e \in E_1$, and $d_e(x) = x$ for $e \in E_2$. Each player i has two pure strategies: $\{h_{i-1}, g_{i-1}\}$ and $\{h_{(i-2) \pmod 3}, h_{i \pmod 3}, g_{i \pmod 3}\}$. The strategy profile in which every player selects his first strategy is a social optimum of cost $(1+\alpha) \cdot 3 + 3 = (2+\alpha) \cdot 3$.

Consider the strategy profile s in which every player chooses his second strategy. We argue that s is a Nash equilibrium. Each player's perceived individual cost is $c_1 = (1-\alpha)(4(1+\alpha)+1) + \alpha(5+4\alpha) \cdot 3$, whereas if a player unilaterally deviates to his first strategy, the new social cost would become $(5+4\alpha) \cdot 3 + 1 - \alpha$. Thus, the player's new perceived individual cost is $c_2 = (1-\alpha)(3(1+\alpha)+2) + \alpha((5+4\alpha) \cdot 3 + 1 - \alpha)$. Because $c_1 = c_2$, s is a Nash equilibrium, of cost $4(1+\alpha) \cdot 3 + 3 = (5+4\alpha) \cdot 3$. We conclude that the price of anarchy is at least $\frac{5+4\alpha}{2+\alpha}$ for $\alpha \in [0, 1]$.

We turn to the pure price of stability of α-altruistic congestion games. Again, an upper bound on the pure price of stability extends to the mixed, correlated and coarse price of stability.

Proposition 11. *The pure price of stability of uniformly α-altruistic linear congestion games is at most $\frac{2}{1+\alpha}$.*

Proof. Let G^α be a uniformly α-altruistic extension of a linear congestion game. It is not hard to verify that G^α is an exact potential game with potential function $\Phi^\alpha(s) = (1-\alpha)\Phi(s) + \alpha C(s)$, where $\Phi(s) =$

$\sum_{e\in E}\sum_{i=1}^{x_e(s)} i$ is Rosenthal's potential function. Observe that

$$\begin{aligned}\Phi^\alpha(s) &= (1-\alpha)\sum_{e\in E}\sum_{i=1}^{x_e(s)} i + \alpha C(s) = \frac{1-\alpha}{2}\sum_{e\in E}(x_e^2(s)+x_e(s)) + \alpha\sum_{e\in E} x_e^2(s)\\ &= \frac{1+\alpha}{2}C(s) + \frac{1-\alpha}{2}\sum_{e\in E} x_e(s).\end{aligned}$$

We therefore have $\frac{1+\alpha}{2}C(s) \leq \Phi^\alpha(s) \leq C(s)$. The claim now follows by using similar arguments as in which proves the claim. □

6 Symmetric Singleton Congestion Games

Symmetric singleton congestion games are an important special case of congestion games. They are defined as $G = (N, E, \{\Sigma_i\}_{i\in N}, \{d_e\}_{e\in E})$: every player chooses one facility (also called *edge*) from $E = \{1, ..., m\}$, and all strategy sets are identical, i.e., $\Sigma_i = E$ for every i. We refer to these games simply as *singleton congestion games* below. In *singleton linear congestion games*, the focus here, delay functions are also assumed to be linear, of the form $d_e(x) = a_e x + b_e$.

6.1 Uniform Altruism

Caragiannis et al. [11] prove the following theorem (stated using the transformation from Remark 1). It shows that the pure price of anarchy does not always increase with the altruism level; the relationship between α and the price of anarchy is thus rather subtle.

Theorem 12 (Caragiannis et al. [11])**.** *The pure price of anarchy of uniformly α-altruistic singleton linear congestion games is $\frac{4}{3+\alpha}$.*

We show that even the mixed price of anarchy (and thus also the robust price of anarchy) will be at least 2 regardless of the altruism levels of the players, by generalizing a result of Lücking et al. [25, Theorem 5.4]. This implies that the benefits of higher altruism in singleton congestion games are only reaped in pure Nash equilibria, and the gap between the pure and mixed price of anarchy increases in α. Also it shows that singleton congestion games constitute a class of games for which the smoothness argument cannot deliver tight bounds.

Proposition 13. *For every $\alpha \in [0, 1]^n$, the mixed price of anarchy for α-altruistic singleton linear congestion games is at least* 2*.*

Proof. Let $m \geq 2$ and consider the instance with player set $\{1, \ldots, m\}$ and facility set $\{1, \ldots, m\}$, with $d_e(x) = x$ ($a_e = 1$ and $b_e = 0$) for each facility e. Denote by s the mixed strategy where each player chooses each link with probability $1/m$. When $\alpha_i = 0$ for every player, s is a mixed Nash equilibrium, and $\mathbf{E}[C(s)] = 2m - 1$ as proved in [25]. The optimum is clearly m, so the price of anarchy of this instance is $2 - 1/m$.

All that is left to show is that s is also a Nash equilibrium under arbitrary altruism levels. By symmetry, it suffices to show that the expected cost of player 1 increases if he deviates to the strategy where he chooses facility 1 with probability 1. Let $s_1^* = 1$. We have

$$\begin{aligned}\mathbf{E}[C_1^\alpha(s_1^*, s_{-1})] &= \mathbf{E}[(1-\alpha_1)C_1(s_1^*, s_{-1}) + \alpha_1 C(s_1^*, s_{-1})]\\ &= (1-\alpha_1)\mathbf{E}[C_1(s_1^*, s_{-1})] + \alpha_1\mathbf{E}[C(s_1^*, s_{-1})].\end{aligned}$$

We already know that $\mathbf{E}[C_1(s_1^*, s_{-1})] \geq \mathbf{E}[C_1(s)]$ because s is a Nash equilibrium when the players are completely selfish, so we are done when we show $\mathbf{E}[C(s_1^*, s_{-1})] \geq \mathbf{E}[C(s)] = 2m - 1$.

For an arbitrary pure strategy profile s, let $X_{i,e}(s')$ be the indicator function that maps to 1 if player i chooses facility e under s', and 0 otherwise. Then it is clear that $C_i(s') = \sum_{e=1}^{m} X_{i,e}(s') d_e(s')$ for $i = 1, \ldots, m$, and $d_e(s) = \sum_{i=1}^{m} X_{i,e}(s')$ for $e = 1, \ldots, m$. So $C_i(s') = \sum_{e,j=1}^{m} X_{i,e}(s') X_{j,e}(s')$. Using this last identity, along with symmetry, independence, and linearity of expectation, the following derivation is easily made (letting $s' = (s_1^*, s_{-1})$):

$$\begin{aligned}
\mathbf{E}[C(s_1^*, s_{-1})] &= \sum_{i=1}^{m} \mathbf{E}[C_i(s')] \\
&= \mathbf{E}[C_1(s')] + (m-1)\mathbf{E}[C_2(s')] \\
&= \mathbf{E}[d_1(s')] + (m-1) \sum_{e,j=1}^{m} \mathbf{E}[X_{2,e}(s') X_{j,e}(s')] \\
&= \sum_{i=1}^{m} \mathbf{E}[X_{i,1}(s')] + (m-1) \left(\sum_{j=1}^{m} \mathbf{E}[X_{2,1}(s') X_{j,1}(s')] + (m-1) \sum_{j=1}^{m} \mathbf{E}[X_{2,2}(s') X_{j,2}(s')] \right) \\
&= \left(1 + (m-1)\frac{1}{m}\right) + (m-1)\left(\frac{1}{m} + \frac{1}{m} + (m-2)\frac{1}{m^2} + (m-1)\left(0 + \frac{1}{m} + (m-2)\frac{1}{m^2}\right)\right) \\
&= 2m - 1.
\end{aligned}$$

□

6.2 Non-Uniform Altruism

We analyze the case when all altruism levels are in $\{0, 1\}$, i.e., each player is either completely altruistic or completely selfish.[7] Then, the system is entirely characterized by the fraction α of altruistic players (which coincides with the average altruism level). The next theorem shows that in this case, too, the pure price of anarchy *improves* with the overall altruism level.

Theorem 14. *Assume that an α fraction of the players are completely altruistic, and the remaining $(1-\alpha)$ fraction are completely selfish. Then, the pure price of anarchy of the altruistic singleton linear congestion game is at most $\frac{4-2\alpha}{3-\alpha}$.*

Let s be a pure Nash equilibrium of G^α and s^* an optimal strategy profile. Again, let $x_e = x_e(s)$ and $x_e^* = x_e(s^*)$. Based on the strategy profile s, we partition the edges in E into sets E_0, E_1:

$$E_1 = \{e \in E : \exists i \in N \text{ with } \alpha_i = 1 \text{ and } s_i = \{e\}\},$$

is the set of edges having at least one altruistic player, while $E_0 = E \setminus E_1$ is the set of edges that are used exclusively by selfish players or not used at all. Let N_1 and N_0 refer to the respective player sets that are assigned to E_1 and E_0. N_1 may contain both altruistic and selfish players, while N_0 consists of selfish players only. Let $k_1 = \sum_{e \in E_1} x_e$ and $k_0 = n - k_1$ denote the number of players in N_1 and N_0, respectively.

The high-level approach of our proof is as follows: We split the total cost $C(s)$ of the pure Nash equilibrium into $C(s) = \gamma C(s) + (1-\gamma) C(s)$ for some $\gamma \in [0, 1]$ such that $\gamma C(s) = \sum_{e \in E_0} x_e d_e(x_e)$ and $(1-\gamma) C(s) = \sum_{e \in E_1} x_e d_e(x_e)$. We bound these two contributions separately to show that

$$\tfrac{3}{4}\gamma C(s) + (1-\gamma) C(s) \le C(s^*). \tag{8}$$

The pure price of anarchy is therefore at most $(\frac{3}{4}\gamma + (1-\gamma))^{-1} = \frac{4}{4-\gamma}$. The bound then follows by deriving an upper bound on γ in Lemma 18.

Lemma 15. *Let s be a pure Nash equilibrium and assume that the delay functions $(d_e)_{e \in E}$ are semi-convex. Then there is an optimal strategy profile s^* such that $x_e(s) \le x_e(s^*)$ for every edge $e \in E_1$.*

[7] This model relates naturally to *Stackelberg scheduling games* (see, e.g., [13]).

Proof. Let s^* be an optimal strategy profile, let x_e denote $x_e(s)$, let x_e^* denote $x_e(s^*)$, and assume that $x_e^* < x_e$ for some $e \in E_1$. Then there is some edge $\bar{e} \in E$ with $x_{\bar{e}}^* > x_{\bar{e}}$. Consider an altruistic player $i \in N_1$ with $s_i = \{e\}$. (Note that i must exist by the definition of E_1.) Because s is a pure Nash equilibrium, player i has no incentive to deviate from e to $\bar{e}$, i.e., $C(\{\bar{e}\}, s_{-i}) \geq C(s)$, or, equivalently,

$$(x_{\bar{e}} + 1)d_{\bar{e}}(x_{\bar{e}} + 1) - x_{\bar{e}}d_{\bar{e}}(x_{\bar{e}}) \geq x_e d_e(x_e) - (x_e - 1)d_e(x_e - 1). \tag{9}$$

Since $x_e^* < x_e$ and $x_{\bar{e}} < x_{\bar{e}}^*$, the semi-convexity of the delay functions implies

$$\begin{aligned}(x_e^* + 1)d_e(x_e^* + 1) - x_e^* d_e(x_e^*) &\leq x_e d_e(x_e) - (x_e - 1)d_e(x_e - 1), & (10)\\ (x_{\bar{e}} + 1)d_{\bar{e}}(x_{\bar{e}} + 1) - x_{\bar{e}}d_{\bar{e}}(x_{\bar{e}}) &\leq x_{\bar{e}}^* d_{\bar{e}}(x_{\bar{e}}^*) - (x_{\bar{e}}^* - 1)d_{\bar{e}}(x_{\bar{e}}^* - 1). & (11)\end{aligned}$$

By combining (9), (10) and (11) and re-arranging terms, we obtain

$$(x_e^* + 1)d_e(x_e^* + 1) + (x_{\bar{e}}^* - 1)d_{\bar{e}}(x_{\bar{e}}^* - 1) \leq x_e^* d_e(x_e^*) + x_{\bar{e}}^* d_{\bar{e}}(x_{\bar{e}}^*).$$

The above inequality implies that by moving a player j with $s_j^* = \{\bar{e}\}$ from $\bar{e}$ to e, we obtain a new strategy profile $s' = (\{e\}, s_{-j}^*)$ of cost $C(s') \leq C(s^*)$. (Note that j must exist because $x_{\bar{e}}^* > x_{\bar{e}} \geq 0$.) Moreover, the number of players on e under the new strategy profile s' increased by one. We can therefore repeat the above argument (with s' in place of s^*) until we obtain an optimal strategy profile that satisfies the claim. □

Note that Lemma 15 implies that at least for singleton congestion games, entirely altruistic players will ensure that Nash equilibria are optimal.

Corollary 16. *The pure price of anarchy of* 1*-altruistic extensions of symmetric singleton congestion games with semi-convex delay functions is* 1.

Henceforth, we assume that s^* is an optimal strategy profile that satisfies the statement of Lemma 15.

Lemma 17. *Define* y^* *as* $y_e^* = x_e^* - x_e \geq 0$ *for every* $e \in E_1$*, and* $y_e^* = x_e^*$ *for all edges* $e \in E_0$*. Then,* $\sum_{e \in E_0} x_e d_e(x_e) \leq \frac{4}{3} \sum_{e \in E} y_e^* d_e(x_e^*)$.

Proof. Consider the game $\bar{G}$ induced by G^α if all k_1 players in N_1 are fixed on the edges in E_1 according to s. Note that all remaining $k_0 = n - k_1$ players in N_0 are selfish. That is, $\bar{G}$ is a symmetric singleton congestion game with player set N_0, edge set E and delay functions $(\bar{d}_e)_{e \in E}$, where $\bar{d}_e(z) = d_e(x_e + z)$ if $e \in E_1$ and $\bar{d}_e(z) = d_e(z)$ for $e \in E_0$. Let $\bar{s}$ be the restriction of s to the players in N_0, and define $\bar{x}$ as $\bar{x}_e = 0$ for $e \in E_1$ and $\bar{x}_e = x_e$ for $e \in E_0$. It is not hard to verify that $\bar{s}$ is a pure Nash equilibrium of the game $\bar{G}$. Let $\bar{s}^*$ be a socially optimum profile for $\bar{G}$, and for each edge e, let $\bar{x}_e^*$ be the total number of players on e under $\bar{S}^*$. Then,

$$\sum_{e \in E_0} x_e d_e(x_e) = \sum_{e \in E} \bar{x}_e \bar{d}_e(\bar{x}_e) \leq \frac{4}{3} \sum_{e \in E} \bar{x}_e^* \bar{d}_e(\bar{x}_e^*) \leq \frac{4}{3} \sum_{e \in E} y_e^* \bar{d}_e(y_e^*) = \frac{4}{3} \sum_{e \in E} y_e^* d_e(x_e^*),$$

where the first inequality follows from Theorem 12 and the second inequality follows from the optimality of $\bar{x}^*$. □

Lemma 18. *It holds that* $\gamma \leq \frac{2n_0}{n+n_0} = \frac{2(1-\alpha)}{2-\alpha}$.

Proof. The claim follows directly from Theorem 12 if $N_1 = \emptyset$. Assume that $N_1 \neq \emptyset$, and let $j \in N_1$ with $s_j = \{\bar{e}\}$. Let $\bar{C}(s) = \sum_{i \in N_0} C_i(s)/k_0$ be the average cost experienced by players in N_0. We first show $C_j(s) \geq \frac{1}{2}\bar{C}(s)$. If $N_0 = \emptyset$, then $C_j(s) \geq \frac{1}{2}\bar{C}(s)$ trivially holds. Suppose that $N_0 \neq \emptyset$, and let $i \in N_0$ with $s_i = \{e\}$. Recall that i is selfish. Because s is a Nash equilibrium, we have

$$C_i(s) = a_e x_e + b_e \leq a_{\bar{e}}(x_{\bar{e}} + 1) + b_{\bar{e}} \leq 2(a_{\bar{e}} x_{\bar{e}} + b_{\bar{e}}) = 2C_j(s).$$

By summing over all k_0 selfish players in N_0, we obtain $C_j(s) \geq \frac{1}{2}\bar{C}(s)$ and thus $\sum_{j \in N_1} C_j(s) \geq \frac{1}{2}k_1\bar{C}(S)$. We have

$$\gamma = \frac{\sum_{i\in N_0} C_i(s)}{\sum_{i\in N_0} C_i(s) + \sum_{j\in N_1} C_j(s)} \leq \frac{k_0\bar{C}(S)}{k_0\bar{C}(S) + \frac{1}{2}k_1\bar{C}(S)} = \frac{2k_0}{n+k_0} \leq \frac{2n_0}{n+n_0},$$

where the last inequality follows because $k_0 \leq n_0$. □

Proof of Theorem 14. Using the above lemmas, we can show that the relation in (8) holds:

$$\begin{aligned}
\frac{3}{4}\gamma C(s) + (1-\gamma)C(s) &= \frac{3}{4}\sum_{e\in E_0} x_e d_e(x_e) + \sum_{e\in E_1} x_e d_e(x_e) \leq \sum_{e\in E} y_e^* d_e(x_e^*) + \sum_{e\in E_1} x_e d_e(x_e) \\
&= \sum_{e\in E} x_e^* d_e(x_e^*) + \sum_{e\in E_1} (x_e d_e(x_e) - x_e d_e(x_e^*)) \leq \sum_{e\in E} x_e^* d_e(x_e^*) = C(s^*),
\end{aligned}$$

where the first inequality follows from Lemma 17 and the last inequality follows from Lemma 15 and because delay functions are monotone increasing. We conclude that the pure price of anarchy is at most

$$\left(\frac{3}{4}\gamma + (1-\gamma)\right)^{-1} = \frac{4}{4-\gamma} \leq \frac{4-2\alpha}{3-\alpha}.$$

The bound now follows from Lemma 18. □

7 General Properties of Smoothness

For the game classes that we analyzed (with the exception of symmetric singleton congestion games), we used (λ,μ,α)-smoothness as our main tool to derive bounds on the price of anarchy. In this section, we provide some general results about (λ,μ,α)-smoothness.

Proposition 19. *Suppose that $\mathcal{G}$ is a class of cost-minimization games equipped with sum-bounded social cost functions. The set $S_{\mathcal{G}} = \{(\lambda,\mu,\alpha) : \forall G \in \mathcal{G},\ G^\alpha$ is (λ,μ,α)-smooth$\}$ is convex.*

Proof. Pick an arbitrary game $G \in \mathcal{G}$. It suffices to show that $S_G = \{(\lambda,\mu,\alpha) : G^\alpha$ is (λ,μ,α)-smooth$\}$ is convex, because the intersection of any collection of convex sets is always convex.

Let $(\lambda_1,\mu_1,\alpha^1), (\lambda_2,\mu_2,\alpha^2) \in S_G$ be two elements in S_G, and pick an arbitrary $\gamma \in [0,1]$. For all pairs (s,s^*) of strategy profiles of G,

$$\begin{aligned}
&\gamma\sum_{i=1}^n (C_i(s_i^*,s_{-i}) + \alpha_i^1(C_{-i}(s_i^*,s_{-i}) - C_{-i}(s))) + \ (1-\gamma)\sum_{i=1}^n (C_i(s_i^*,s_{-i}) + \alpha_i^2(C_{-i}(s_i^*,s_{-i}) - C_{-i}(s))) \\
\leq\ &\gamma(\lambda_1 C(s^*) + \mu_1 C(s)) + (1-\gamma)(\lambda_2 C(s^*) + \mu_2 C(s)).
\end{aligned}$$

By rewriting both sides of the above inequality, we obtain

$$\begin{aligned}
&\sum_{i=1}^n (C_i(s_i^*,s_{-i}) + (\gamma\alpha_i^1 + (1-\gamma)\alpha_i^2)(C_{-i}(s_i^*,s_{-i}) - C_{-i}(s))) \\
&\qquad \leq (\gamma\lambda_1 + (1-\gamma)\lambda_2)C(s^*) + (\gamma\mu_1 + (1-\gamma)\mu_2)C(s).
\end{aligned}$$

We conclude that G is $(\gamma(\lambda_1,\mu_1,\alpha^1) + (1-\gamma)(\lambda_2,\mu_2,\alpha^2))$-smooth. Therefore, S_G is convex. □

A natural question to ask is whether the robust price of anarchy is also a convex function of α. This turns out not to be the case. For instance, the robust price of anarchy for uniformly α-altruistic congestion games is $\frac{5+4\alpha}{2+\alpha}$ (see Section 5), which is a non-convex function. However, we can prove a somewhat weaker statement: For a subset $S \subseteq \mathbb{R}^n$, we call a function $f : S \to \mathbb{R}$ *quasi-convex* iff $f(\gamma x + (1-\gamma)y) \leq \max\{f(x), f(y)\}$ for all $\gamma \in [0,1]$.

Theorem 20. *Let $\mathcal{G}$ be a class of games equipped with sum-bounded social cost functions. Then $RPoA_{\mathcal{G}}(\alpha)$ is a quasi-convex function of α.*

Proof. Let $G \in \mathcal{G}$. We show that for any $\alpha^1, \alpha^2 \in \mathbb{R}^n$ and $\gamma \in [0, 1]$,

$$\text{RPoA}(\gamma\alpha^1 + (1-\gamma)\alpha^2) \leq \max\{\text{RPoA}(\alpha^1), \text{RPoA}(\alpha^2)\}.$$

Let $(\epsilon_1, \epsilon_2, \ldots)$ be a decreasing sequence of positive real numbers that tends to 0. Moreover, let

$$((\lambda_{1,1}, \mu_{1,1}, \alpha^1), (\lambda_{1,2}, \mu_{1,2}, \alpha^1), \ldots) \quad \text{and} \quad ((\lambda_{2,1}, \mu_{2,1}, \alpha^2), (\lambda_{2,2}, \mu_{2,2}, \alpha^2), \ldots)$$

be sequences of elements in S_G (where S_G is as defined in the proof of Proposition 19) such that

$$\text{RPoA}(\alpha^1) + \epsilon_j = \tfrac{\lambda_{1,j}}{1-\mu_{1,j}} \quad \text{and} \quad \text{RPoA}(\alpha^2) + \epsilon_j = \tfrac{\lambda_{2,j}}{1-\mu_{2,j}}$$

for all j. By Proposition 19, we know that for all j,

$$\begin{aligned}\sum_{i=1}^{n}&(C_i(s_i^*, s_{-i}) + (\gamma\alpha_i^1 + (1-\gamma)\alpha_i^2)(C_{-i}(s_i^*, s_{-i}) - C_{-i}(s)))\\ &\leq \gamma(\lambda_{1,j}C(s^*) + \mu_{1,j}C(s)) + (1-\gamma)(\lambda_{2,j}C(s^*) + \mu_{2,j}C(s))\\ &\leq \max\{\lambda_{1,j}C(s^*) + \mu_{1,j}C(s), \lambda_{2,j}C(s^*) + \mu_{2,j}C(s)\}.\end{aligned}$$

Hence,

$$\text{RPoA}(\gamma\alpha^1 + (1-\gamma)\alpha^2) \leq \max\left\{\frac{\lambda_{1,j}}{1-\mu_{1,j}}, \frac{\lambda_{2,j}}{1-\mu_{2,j}}\right\} \leq \max\{\text{RPoA}(\alpha^1), \text{RPoA}(\alpha^2)\} + \epsilon_j,$$

for all j. By taking the limit as j goes to infinity, we conclude $\text{RPoA}(\gamma\alpha^1 + (1-\gamma)\alpha^2) \leq \max\{\text{RPoA}(\alpha^1), \text{RPoA}(\alpha^2)\}$, which proves the claim. ☐

The quasi-convexity of $\text{RPoA}_{\mathcal{G}}$ implies:

Corollary 21. *The points α that minimize $RPoA_{\mathcal{G}}(\alpha)$ on the domain $[0,1]^n$ form a convex set. The set of points α that maximize $RPoA_{\mathcal{G}}(\alpha)$ on the domain $[0,1]^n$ includes at least one point that is a 0-1 vector.*

8 Conclusions and Future Work

One might not expect that there are games in which the price of anarchy is greater than 1 when $\alpha = \mathbf{1}$. This phenomenon is a lot less surprising when approached from a local search point-of-view, as this is only equivalent to saying that there exist local optima in the objective function C with respect to the neighborhood set obtained by taking all strategies obtained by single-player deviations from a given strategy profile s. Nevertheless, it still seems to us rather surprising that the price of anarchy can get worse when the altruism level α gets closer to $\mathbf{1}$. This phenomenon has been observed before, in [11]. The fact that the price of anarchy does not *necessarily* get worse in all cases is exemplified by our analysis of the pure price of anarchy in symmetric singleton congestion games.

The most immediate future directions include analyzing singleton congestion games with more general delay functions than linear ones. While the price of anarchy of such functions increases (e.g., the price of anarchy for polynomials increases exponentially in the degree [4, 14]), this also creates room for potentially larger reductions due to altruism. Similarly, the characterization of the robust price of anarchy of altruistic congestion games with more general delay functions (e.g., polynomials) is left for future work.

For games where the smoothness argument cannot give tight bounds, would a refined smoothness argument like local smoothness in [34] work? For symmetric singleton congestion games, this seems unlikely, as the price of anarchy bounds are already different between pure and mixed Nash equilibria. It is also worth trying to apply the smoothness argument or its refinements to analyze the price of anarchy for other dynamics in other classes of altruistic games, for example, (altruistic) network vaccination games [12], which are known to not always possess pure Nash equilibria, or to find examples to see why smoothness-based arguments do not work.

We have seen that the impact of altruism depends on the underlying game. It would be nice to identify general properties that enable to predict whether a given game suffers from altruism or not. What is it that makes valid utility game invariant to altruism? Furthermore, what kind of "transformations" (not just altruistic extensions) might be applied to a strategic game such that the smoothness approach can still be adapted to give (tight) bounds? More generally, while the existence of pure Nash equilibria has been shown for singleton and matroid congestion games with player-specific latency functions [1, 26], the price of anarchy (for pure Nash equilibria or more general equilibrium concepts) has not yet been addressed. Studying the price of anarchy in such a general setting (in which our setting with altruism can be embedded) by either smoothness-based techniques or other methods is undoubtedly intriguing.

References

[1] H. Ackermann, H. Röglin, and B. Vöcking. Pure Nash equilibria in player-specific and weighted congestion games. In *Proc. 2th Workshop on Internet and Network Economics)*, 2006.

[2] E. Anshelevich, A. Dasgupta, J. Kleinberg, E. Tardos, T. Wexler, and T. Roughgarden. The price of stability for network design with fair cost allocation. In *Proc. 45th Symposium on Foundations of Computer Science*, 2004.

[3] R. J. Aumann. Subjectivity and correlation in randomized strategies. *J. Mathematical Economics*, 1(1):67–96, 1974.

[4] B. Awerbuch, Y. Azar, and A. Epstein. Large the price of routing unsplittable flow. In *Proc. 37th Annual ACM Symposium on Theory of Computing*, pages 57–66, 2005.

[5] M. Babaioff, R. Kleinberg, and C. H. Papadimitriou. Congestion games with malicious players. *Games and Economic Behavior*, 67(1):22–35, 2009.

[6] T. Bergstrom. Systems of benevolent utility functions. *J. Public Economic Theory*, 1(1):71–100, 1999.

[7] A. Blum, E. Even-Dar, and K. Ligett. Routing without regret: On convergence to Nash equilibria of regret-minimizing algorithms in routing games. In *Proc. 25th Annual ACM Symposium on Principles of Distributed Computing*, pages 45–52, 2006.

[8] A. Blum, M. T. Hajiaghayi, K. Ligett, and A. Roth. Regret minimization and the price of total anarchy. In *Proc. 40th Annual ACM Symposium on Theory of Computing*, pages 373–382, 2008.

[9] M. Bradonjic, G. Ercal-Ozkaya, A. Meyerson, and A. Roytman. On the price of mediation. In *Proc. 11th ACM Conference on Electronic Commerce*, 2009.

[10] F. Brandt, T. Sandholm, and Y. Shoham. Spiteful bidding in sealed-bid auctions. In *Proc. 20th International Joint Conference on Artifical intelligence*, pages 1207–1214, 2007.

[11] I. Caragiannis, C. Kaklamanis, P. Kanellopoulos, M. Kyropoulou, and E. Papaioannou. The impact of altruism on the efficiency of atomic congestion games. In *Proc. 5th Symposium on Trustworthy Global Computing*, 2010.

[12] P.-A. Chen, M. David, and D. Kempe. Better vaccination strategies for better people. In *Proc. 12th ACM Conference on Electronic Commerce*, 2010.

[13] P.-A. Chen and D. Kempe. Altruism, selfishness, and spite in traffic routing. In *Proc. 10th ACM Conference on Electronic Commerce*, pages 140–149, 2008.

[14] G. Christodoulou and E. Koutsoupias. The price of anarchy of finite congestion games. In *In Proc. 37th Annual ACM Symposium on Theory of Computing*, 2005.

[15] J. Elias, F. Martignon, K. Avrachenkov, and G. Neglia. Socially-aware network design games. In *Proc. INFOCOM 2010*, pages 41–45, 2010.

[16] J. Geanakoplos, D. Pearce, and E. Stacchetti. Psychological games and sequential rationality. *Games and Economic Behavior*, 1(1):60–79, March 1989.

[17] H. Gintis, S. Bowles, R. Boyd, and E. Fehr. *Moral Sentiments and Material Interests : The Foundations of Cooperation in Economic Life.* MIT Press, 2005.

[18] J. Hannan. Approximation to bayes risk in repeated plays. *Contributions to the Theory of Games*, 3:97–193, 1957.

[19] A. Hayrapetyan, E. Tardos, and T. Wexler. The effect of collusion in congestion games. In *Proc. 38th Annual ACM Symposium on Theory of Computing*, pages 89–98, 2006.

[20] M. Hoefer and A. Skopalik. Altruism in atomic congestion games. In *Proc. 17th European Symposium on Algorithms*, 2009.

[21] G. Karakostas and A. Viglas. Equilibria for networks with malicious users. *Math. Program.*, 110:591–613, 2007.

[22] E. Koutsoupias and C. Papadimitriou. Worst-case equilibria. In *Proc. 16th Annual Symposium on Theoretical Aspects of Computer Science*, pages 404–413, 4–6 1999.

[23] John Ledyard. Public goods: A survey of experimental resesearch. In J. Kagel and A. Roth, editors, *Handbook of Experimental Economics*, pages 111–194. Princeton University Press, 1997.

[24] D. K. Levine. Modeling altruism and spitefulness in experiments. *Review of Economic Dynamics*, 1(3):593–622, July 1998.

[25] T. Lücking, M. Mavronicolas, B. Monien, and M. Rode. A new model for selfish routing. *Theoretical Computer Science*, 406:187–206, October 2008.

[26] I. Milchtaich. Congestion games with player-specific payoff functions. *Games and Economic Behavior*, 13(1):111–124, 1996.

[27] T. Moscibroda, S. Schmid, and R. Wattenhofer. When selfish meets evil: Byzantine players in a virus inoculation game. In *Proc. 25th Annual ACM Symposium on Principles of Distributed Computing*, pages 35–44, 2006.

[28] U. Nadav and T. Roughgarden. The limits of smoothness: A primal-dual framework for price of anarchy bounds. In *Proc. 6th Workshop on Internet and Network Economics*, pages 319–326, 2010.

[29] N. Nisan, T. Roughgarden, É. Tardos, and V. V. Vazirani, editors. *Algorithmic Game Theory.* Cambridge University Press, 2007.

[30] M. Rabin. Incorporating fairness into game theory and economics. *American Economic Review*, 83(5):1281–1302, December 1993.

[31] A. Roth. The price of malice in linear congestion games. In *Proc. 4th Workshop on Internet and Network Economics*, pages 118–125, 2008.

[32] T. Roughgarden. *Selfish Routing and the Price of Anarchy.* MIT Press, 2005.

[33] T. Roughgarden. Intrinsic robustness of the price of anarchy. In *Proc. 41st Annual ACM Symposium on Theory of Computing*, pages 513–522, 2009.

[34] T. Roughgarden and F. Schoppmann. Local smoothness and the price of anarchy in atomic splittable congestion games. In *Proc. 22nd ACM-SIAM Symposium on Discrete Algorithms*, 2011.

[35] A. Vetta. Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions. In *Proc. 43rd Symposium on Foundations of Computer Science*, 2002.

[36] H. P. Young. *Strategic Learning and its Limits.* Oxford University Press, 1995.