On Strong Equilibria and Improvement Dynamics in Network Creation Games

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Abstract. We study strong equilibria in network creation games. These form a classical and well-studied class of games where a set of players form a network by buying edges to their neighbors at a cost of a fixed parameter α. The cost of a player is defined to be the cost of the bought edges plus the sum of distances to all the players in the resulting graph. We identify and characterize various structural properties of strong equilibria, which lead to a characterization of the set of strong equilibria for all α in the range $(0, 2)$. For $\alpha > 2$, Andelman et al. (2006) prove that a star graph in which every leaf buys one edge to the center node is a strong equilibrium, and conjecture that in fact *any* star is a strong equilibrium. We resolve this conjecture in the affirmative. Additionally, we show that when α is large enough ($\geq 2n$) there exist non-star trees that are strong equilibria. For the strong price of anarchy, we provide precise expressions when α is in the range $(0, 2)$, and we prove a lower bound of $3/2$ when $\alpha \geq 2$. Lastly, we aim to characterize under which conditions (coalitional) improvement dynamics may converge to a strong equilibrium. To this end, we study the (coalitional) finite improvement property and (coalitional) weak acyclicity property. We prove various conditions under which these properties do and do not hold. Some of these results also hold for the class of pure Nash equilibria.

1 Introduction

The Internet is a large-scale network that has emerged mostly from the spontaneous, distributed interaction of selfish agents. Understanding the process of creating of such networks is an interesting scientific problem. Insights into this process may help to understand and predict how networks emerge, change, and evolve. This holds in particular for social networks.

The field of game theory has developed a large number of tools and models to analyze the interaction of many independent agents. The Internet and many other networks can be argued to have formed through interaction between many strategic agents. It is therefore natural to use game theory to study the process of network formation. Indeed, this has been the subject of study in many research papers, e.g. [15,1,6,2,16,14,22], to mention only a few of them.

We focus here on the classical network creation model of [15], which is probably the class of network formation game that is most prominently studied by algorithmic game theorists. This model stands out due to its simplicity and elegance: It is simply defined as a game on n players, where each player may choose an arbitrary set of edges that connects herself to a subset of other players, so that a graph forms where the vertices are the players. Buying any edge costs a fixed amount $\alpha \in \mathbb{R}$, which is the same for every player. Now, the cost of a player is defined as the total cost of set of edges she bought, plus the sum of distances to all the other players in the graph. A network creation game is therefore determined by two parameters: α and n.

Another reason for why these network creation games are an ineresting topic of study, are the surprisingly challenging questions that emerge from this simple class of games. For example, it is (as of writing) unknown whether the *price of anarchy* of these network creation games is bounded by a constant, where the term price of anarchy is defined as the factor by which the total cost of a pure Nash equilibrium is away from the minimum possible total cost [20,21].

In the present work, we study *strong equilibria*, which are a refinement of the pure Nash equilibrium solution concept. Strong equilibria are defined as pure Nash equilibria that are resilient against strategy changes that are made collectively by arbitrary *sets* of players, in addition to strategy changes that are made by *individual* players (see [5]). Generally, such an equilibrium may not exist, since this is already the case for pure Nash equilibria. On the other hand, in case they do exist, then strong equilibria are extremely robust, and they are likely to describe the final outcome of a game in case they are, in a realistic sense, “easy to attain” for the players. Fortunately, as [4] points out, in network creation games, strong equilibria are guaranteed to exist except in a very limited number of cases. The combination of the facts that strong equilibria are robust, and are almost always guaranteed to exist, calls for a detailed study of these equilibria in network creation games, which is what we do in the present work.

We provide in this paper a complete characterization of the set of all strong equilibria for $\alpha \in (0, 2)$. Moreover, for $\alpha > 2$ we prove in the affirmative the conjecture of [4] that any strategy profile that forms a star graph (i.e., a tree of depth 1) is a strong equilibrium. We also show that for large enough α (namely, for $\alpha \geq 2n$), there exist strong equilibria that result in trees that are not stars.

The price of anarchy restricted to strong equilibria is called the *strong price of anarchy*. This notion was introduced in [4], where also the strong price of anarchy of network creation games was studied first. The authors prove there that the strong price of anarchy is at most 2. We contribute to the understanding of the strong price of anarchy by providing a sequence of examples of strong equilibria where the strong price of anarchy converges to $3/2$, thereby providing the first non-trivial lower bound (to the best of our knowledge).

Regarding the reachability and the likelihood for the players to actually attain a strong equilibrium, we study the question whether they can be reached by *response dynamics*, i.e., the process where we start from any strategy profile, and we repeatedly let a player or a set of players make a change of strategies that is beneficial for each player in the set, i.e., decreases their cost. In particular, we are interested in whether network creation games posess the *coalitional finite improvement property* (that is: whether such response dynamics are guaranteed to result in a strong equilibrium), and the *coalitional weak acyclicity property* (that is: whether there exists a sequence of coalitional strategy changes that ends in a strong equilibrium when starting from any strategy profile). We prove various conditions under which these properties are satisfied. Roughly, we show that coalitional weak acyclicity holds when $\alpha \in (0, 1]$ or when starting from a strategy profile that forms a tree (for $\alpha \in (0, n/2]$), but that the coalitional finite improvement property is unfortunately not satisfied for any α. Some of these results hold for pure Nash equilibria as well.

1.1 Our Contributions

A key publication that is strongly related to our work is [4], where the authors study the existence of strong equilibria in network creation games. The authors prove that the strong price of anarchy of network creation games does not exceed 2 and provide insights into the structure and existence of strong equilibria. This is to the best of our knowledge the only paper studying strong equilibria in network creation games. Let us therefore summarize how the present paper complements and contributes to the results in [4]: First, we provide additional results on the strong equilibrium structure, such that together with the results from [4] we obtain a characterization of strong equilibria for $\alpha \in (0, 2)$. Furthermore, in [4] it was conjectured that all strategy profiles that form a star (and such that no edge is bought by two players at the same time) are strong equilibria. We answer this conjecture positively. Because [4] does not provide examples of strong equilibria that are not stars (for $\alpha > 2$), this may suggest the conjecture that *all* strong equilibria form a star for $\alpha > 2$. We show however that the latter is not true: We provide a family of examples of strong equilibria which form trees of diameter four (hence, not stars). More interestingly, the latter sequence of examples has a price of anarchy that converges to $3/2$, thereby providing (again, to the best of our knowledge) the first non-trivial lower bound on the strong price of anarchy. Related to this set of results, we want to mention the following interesting open questions for future research: (i.) What is the exact strong price of anarchy of the class of network creation games? Our work shows that it must lie in the interval $[3/2, 2]$. (ii.) Does there exist a non-star strong equilibrium for $\alpha \in (2, 2n)$? (iii.) Do there exist strong equilibria that form trees of arbitrarily high diameter, and do there exist strong equilibria that are not trees?

A second theme of our paper is to investigate under which circumstances the coalitional finite improvement and coalitional weak acyclicity properties are satisfied, as satisfying those properties contribute to the credibility of strong equilibria as a realistic solution concept. We show to this end that coalitional weak acyclicity always holds for $\alpha \in (0, 1]$ and holds for $\alpha \in (1, n/2)$ in case the starting strategy profile is a tree. We prove on the negative side that for all α there exists a number of players n such that the coalitional finite improvement property does not hold. The only special case for which we manage to establish existence of the coalitional finite improvement property is for $n = 3$ and $\alpha > 1$. With regard to convergence of response dynamics to strong equilibria, an interesting question that we leave open is whether the coalitional weak acyclicity property holds for $\alpha > n/2$, and for $\alpha \in (1, n/2)$ when starting at non-tree strategy profiles. We will see throughout that some of our results on these properties also hold for the set of pure Nash equilibria.

An overview of results is summarized in the tables below. Table 1 provides an overview for our characterization and structure theorems for strong equilibria, Table 2 shows our bounds on the strong price of anarchy, and Table 3 shows our results on the finite improvement and weak acyclicity properties of network creation games. Due to space constraints, the proofs of many of our results have been omitted and will be published in a full version of the paper.

	$\alpha \in (0, 1)$	$\alpha = 1$	$\alpha \in (1, 2)$	$\alpha \geq 2$
strong equilibria	Characterized (in [4])	Characterized (Theorem 1)	Characterized (Proposition 1)	Every star is a strong equilibrium (Theorem 2), existence of non-star strong equilibria (Theorem 3)

Table 1. Overview of strong equilibria characterization results and structural results.

	$\alpha \in (0, 1)$	$\alpha = 1$	$\alpha \in (1, 2)$	$\alpha \geq 2$
strong price of anarchy	1 (Trivial)	$10/9$ if $n \leq 4$ and $(3n + 2)/3n$ if $n \geq 5$ (Theorem 4)	$(2\alpha + 8)/(3\alpha + 6)$ if $n = 3$, and $(4\alpha+16)/(6\alpha+12)$ if $n = 4$ (Proposition 3)	At least $3/2$ (Theorem 5) and at most 2 [4]

Table 2. Overview of bounds on the strong price of anarchy .

	$\alpha \in (0, 1)$	$\alpha = 1$	$\alpha \in (1, 2)$	$\alpha = 2$	$\alpha > 2$
c-FIP	Negative (Lemma 6)	Negative (Lemma 6)	Negative (Lemma 6)	Negative (Lemma 6)	Negative (in [9])
			Positive for $n = 3$ (Lemma 4)		
c-weak acyclicity	Positive (Corollary of Lemma 8)	Positive (Proposition 9)	Positive with respect to trees for $\alpha \in (1, n/2)$ (Lemma 11)		

Table 3. Summary of results on the c-FIP and c-weak acyclicity of network creation games.

2 Related Literature

We discussed already extensively the works [4] and [15]. The latter is the article in which network creation games were first defined. Moreover, [15] conjectured that there exists an $A \in \mathbb{R}_{\geq 0}$ such that all non-transient equilibria (where *transience* stands for a particular notion of instability) are trees for $\alpha \geq A$.

This conjecture was subsequently disproved by [1], where the authors construct non-tree equilibria for abitrarily high α. These equilbiria are *strict* (i.e., for no player there is a deviation that keeps her cost unchanged) and therefore non-transient, and their construction uses finite affine planes. In this paper, the authors moreover show that the price of anarchy is constant for $\alpha \leq \sqrt{n}$ and for $\alpha \geq 12n \log n$, as for the second case they prove that any pure equilibrium is a tree. In [27], the latter bound was improved, as it was shown there that for $\alpha \geq 273n$ all pure equilibria are trees. Later on, in [24], this was further improved by showing that it even holds for $\alpha \geq 65n$. Very recently, in [3], further progress has been made in this direction by showing that every pure Nash equilibrium is a tree already when $\alpha > 17n$, and that the price of anarchy is bounded by a constant for $\alpha > 9n$. In [12], some constant bounds on the price of anarchy were improved, and it was shown that for $\alpha \leq n^{1-\epsilon}$ the price of anarchy is constant, for all $\epsilon \geq 0$. It remains an open question whether the price of anarchy is constant for all $\alpha \in \mathbb{R}_{\geq 0}$. In particular, the best known bound on the price of anarchy for $\alpha \in [n^{1-\epsilon}, 9n]$ is $2^{O(\sqrt{\log n})}$, shown in [12]. For all other choices of α the price of anarchy is known to be constant. The master's thesis [25] provides some simplified proofs for some of the above facts, and proves that if an equilibrium graph has bounded degree, then the price of anarchy is bounded by a constant. It also studies some related computational questions.

Many other variants of network creation games have been considered as well. A version where disconnected players incur a finite cost rather than an infinite one was studied in [9]. In [1], a version is introduced where the distance cost of a player i to another player j is weighted by some number w_{ij}. A special case of this weighted model was proposed in [26]. The paper [12] introduces a version of the game where the distance cost of a player is defined the *maximum* distance from i to any other player (instead of the sum of distances), and studies the price of anarchy for these games. Further results on those games can be found in [27]. Another natural variant of a cost sharing game is one where both endpoints of an edge can contribute to its creation, as proposed in [26], or must share its creation cost equally as proposed in [11] and further investigated in [12]. In [6], a version of the game is studied where the edges are directed, and the distance of a player i to another player j is the minimum length of a directed path from i to j. The literature on these games and generalizations thereof (see e.g., [14,13,8]) concerns existence of equilibria and the properties of response dynamics. See [7,16,18,17] for another undirected network creation model and properties of pure equilibria in those models. Further, in the very recent paper [10], a variant of network creation games is studied where the cost of buying an edge to a player is proportional to the number of neighbors of that player.

In [2], the authors analyze the outcomes of the game under the assumption that the players consider deviations by swapping adjacent edges. Better response dynamics under this assumption have been studied in [22]. A modified version of this model is introduced in [26], where players can only swap their *own* edges. The authors prove some structural results on the pure equilibria that can then arise. Furthermore, in [23] the deviation space is enriched by allowing the players to *add* edges, and various price of anarchy type bounds are established under this assumption. In [19], the dynamics of play in various versions of network creation games are further investigated.

3 Preliminaries

A *network creation game* Γ is a game played by $n \geq 3$ players where the strategy set of $\mathcal{S}_i$ of a player $i \in [n] = \{1, \ldots, n\}$ is given by $\mathcal{S}_i = \{s : s \subseteq [n] \setminus \{i\}\}$. That is, each player chooses a subset of other players. Let $\mathcal{S} = \times_{i \in [n]} \mathcal{S}_i$ be the strategy profiles of Γ and for a subset $K \subseteq [n]$ of players let $\mathcal{S}_K = \times_{i \in K} \mathcal{S}_i$. Given a strategy profile $s \in \mathcal{S}$, we define

$G(s)$ as the undirected graph with vertex set $[n]$ and edge set $\{\{i, j\} : j \in s_i \vee i \in s_j\}$. For a graph G on vertex set $[n]$, we denote by $d_G(i, j)$ the length of the shortest path from i to j in G (and we define the distance between two disconnected vertices as infinity).

The cost of player i under s is given by $c_i(s) = c_i^b(s_i) + c_i^d(s)$, where $c_i^b(s_i) = \alpha|s_i|$ is referred to as the *building cost*, $\alpha \in \mathbb{R}_{\geq 0}$ is a player-independent constant, and $c_i^d(s) = \sum_{j=1}^n d_{G(s)}(i, j)$ is referred to as the *distance cost*. The interpretation given to this game is that the players buy edges to other players and that creates a network. Buying a single edge costs α. The shortest distance $d_{G(s)}(i, j)$ to each other player j is furthermore added to the cost of a player i. We denote a network creation game by the pair (n, α)

For a strategy profile $s \in \mathcal{S}$ let $d(s) = \sum_i c_i^d(s)$. The social cost of strategy profile s, denoted $C(s)$, is defined as the sum of all individual costs: $C(s) = \sum_{i\in[n]} c_i(s) = \alpha \sum_i |s_i| + d(s)$.

We study the *strong equilibria* of this game. A *strong equilibrium* of an n-player cost minimization game Γ with strategy profile set $\mathcal{S} = \times_{i=1}^n \mathcal{S}_i$ is an $s \in \mathcal{S}$ such that for all $K \subseteq [n]$ and for all $s'_K \in \mathcal{S}_K$ there exists a player $i \in K$ such that, $c_i(s) \leq c_i(s'_K, s_{-K})$, where c_i is the cost function of player i and (s'_K, s_{-K}) denotes the vector obtained from s by replacing the $|K|$ elements at index set K with the elements s'_K. (A *pure Nash equilibrium* is a strategy profile that satisfies the latter condition only for singleton K.) Strong equilibria are guaranteed to exist in almost all network creation games, as we will explain later.

We are interested in determining the *strong price of anarchy* [4]. The *strong price of anarchy* of a network creation game Γ is the ratio $\mathrm{PoA}(\Gamma) = \max\{C(s)/C(s^*) : s \in \mathrm{SE}\}$, where s^* is a *social optimum*, i.e., a strategy profile that minimizes the social cost. Furthermore SE is the set of strong equilibria of the game.

A strategy profile s is called *rational* if there is no player pair $i, j \in [n]$ such that $j \in s_i$ and $i \in s_j$. It is clear that all pure Nash equilibria (and thus all strong equilibria) of any network creation game are rational, as are all the social optima. When s is a rational strategy profile, the social cost can be written as $C(s) = \alpha|E(G(s))| + d(s)$, where $E(G(s))$ denotes the edge set of the graph $G(s)$.

We write $\deg_{G(s)}(i)$ to denote the degree of player i in graph $G(s)$, and we denote by $\mathrm{diam}(G(s)))$ the diameter of $G(s)$. We define the *free-riding* function $f : \mathcal{S} \times [n] \to \mathbb{N}$ by the formula $f(s, i) = \deg_{G(s)}(i) - |s_i|$. For any strategy profile $s \in \mathcal{S}$ we have the following lower bound for the cost of player i,

$$c_i(s) \geq 2n - 2 - \deg_{G(s)}(i) + |s_i|\alpha = 2n - 2 - f(s, i) + |s_i|(\alpha - 1). \tag{1}$$

Moreover, we see that in case s is rational,

$$\sum_{i\in[n]} |s_i| = |E| = \sum_{i\in[n]} f(s, i). \tag{2}$$

Graph theory notions. We define an *n-star* to be a tree of n vertices with diameter 2, i.e., it is a tree where one vertex is connected to all other vertices. It is straightforward to verify that (1) is tight when $G(s)$ is an n-star, and (more generally) when $G(s)$ has diameter at most 2. We denote by K_n the complete undirected graph on vertex set $[n]$. We denote by C_n the undirected cycle on vertex set $[n]$. We denote by P_n the undirected path on vertex set $[n]$. Lastly, we define a *centroid* vertex of a tree $T = (V, E)$ as a vertex $v \in V$ that minimizes $\max\{|V_i| : (V_i, E_i) \in C_{T-v}\}$, where C_{T-v} denotes the set of connected components of the subgraph of T induced by $V \setminus \{v\}$.

Coalitional improvement dynamics. A sequence of strategy profiles $(s^1, s^2, \ldots)$ is called a *path* if for every $k > 1$ there exists a player $i \in [n]$ such that $s^k = (s'_i, s^{k-1}_{-i})$. We call a path an *improvement path* if for all $k > 1$ holds $c_i(s^k) < c_i(s^{k-1})$ where i is the player who deviated from s^{k-1}. We say that it is an *improvement cycle* if additionally there exists a constant T such that $s^{k+T} = s^k$ for all $k \geq 1$. A sequence of strategies $(s^1, s^2, \ldots)$ is called a *best response improvement path* if for all $k > 1$ and all i such that $s^k_i \neq s^{k-1}_i$ we have $c_i(s^k) < c_i(s^{k-1})$ and there is no $s'_i \in \mathcal{S}_i$ such that $c_i(s'_i, s^k_{-i}) < c_i(s^k)$ (that is: s^k_i is a *best response* to s^{k-1}_{-i}). A sequence of strategies $(s^1, s^2, \ldots)$ is called a *coalitional improvement path* if for all $k > 1$ and all i such that $s^k_i \neq s^{k-1}_i$ we have $c_i(s^k) < c_i(s^{k-1})$.

A game has the *(coalitional) finite improvement property ((c-)FIP)* if every (coalitional) improvement path is finite. A game has *finite best response property (FBRP)* if every best response improvement path is finite. We call a game *(c-)weakly acyclic* if for every $s \in \mathcal{S}$ there exists a finite (coalitional) improvement path starting from s. Lastly, we call a network creation game *(c-)weakly acyclic with respect to a class of graphs $\mathcal{G}$* if for every $s \in \mathcal{S}$ such that $G(s) \in \mathcal{G}$, there exists a (coalitional) finite improvement path starting from s.

4 Structural Properties of Strong Equilibria

We provide in this section various results that imply a full characterization of strong equilibria for $\alpha \in (0, 2)$, and we resolve a conjecture of [4] by showing that any rational strategy $s \in \mathcal{S}$ such that $G(s)$ is a star is a strong equilibrium for all $\alpha \geq 2$. Moreover, we give a family of examples of strategy profiles that form trees of diameter 4 (hence do not form stars) and are strong equilibria when $\alpha \geq 2n$. First, for $\alpha \in (0, 1)$ the strong equilibrium set is straighforward to derive, as has been pointed out in [4]: in this case a strategy profile is a strong equilibrium if and only if it is rational and forms the complete graph. It is easy to see that this characterization also holds for the set of Nash equilibria.

For $\alpha = 1$, the situation is more complex. First, we can show that the following lemma holds for all $\alpha < 2$.

Lemma 1. *Fix $\alpha < 2$ and suppose that $s \in \mathcal{S}$ is a strong equilibrium. For each sequence of players $(i_0, i_1, \ldots, i_k = i_0)$ such that $k \geq 3$ in $G(s)$ there exists an $t \in \{0, \ldots, k-1\}$ such that $(i_t, i_{t+1}) \in E(G(s))$. In other words, the complement of $G(s)$ is a forest.*

Therefore, if $\alpha < 2$ and $s \in \mathcal{S}$ is a strong equilibrium, then there is no independent set of size 3 in $G(s)$. Also, if $\alpha < 2$ and $|V| \geq 4$, then a strategy profile $s \in \mathcal{S}$, such that $G(s)$ is a star is not a strong equilibrium. Since when $\alpha \in [1, 2)$, a rational strategy profile that forms a star is a Nash equilibrium, this implies that the pure Nash equilibria and strong equilibria do not coincide.

In order to characterize the strong equilibria for $\alpha = 1$, we first provide a characterization of the pure Nash equilibria.

Lemma 2. *For $\alpha = 1$, a strategy profile $s \in \mathcal{S}$ is a Nash equilibrium if and only if s is rational and $G(s)$ has diameter at most 2.*

The following theorem then characterizes the set of strong equilibria for $\alpha = 1$.

Theorem 1. *For $\alpha = 1$, a strategy profile $s \in \mathcal{S}$ is a strong equilibrium if and only if s is rational, $G(s)$ has diameter at most 2, and the complement of $G(s)$ is a forest.*

For $\alpha \in (1, 2)$, it was shown in [4] that strong equilibria do not exist for $n \geq 5$. It can be shown that for $n = 3$ the set of strong equilibria are the rational strategy profiles that form the 3-star. (Hence, all pure Nash equilibria are strong equilibria in this case). For $n = 4$ we observe that the only strong equilibria are those that form the cycle on 4 vertices such that every player buys exactly one edge. Thus, the following proposition completes our characterization of strong equilibria for $\alpha \in (1, 2)$.

Proposition 1. *Let $\alpha \in (1, 2)$ and let $s \in \mathcal{S}$. Then: (i.) If $n = 3$, strategy profile s is a strong equilibrium if and only if s is rational and $G(s)$ is a 3-star. (ii.) If $n = 4$, strategy profile s is a strong equilibrium if and only if s is rational, $|s_i| = 1$ for all i, and $G(s)$ is a cycle. (iii.) If $n \geq 5$, s is not a strong equilibrium.*

Next, we prove the following conjecture of [4].

Theorem 2. *Let $\alpha \geq 2$ and $s \in \mathcal{S}$. If s is rational and $G(s)$ is a star, then s is a strong equilibrium.*

The proof of this theorem relies on two lemmas. The first lemma provide bounds on the free-riding function of player sets who manage to deviate profitably, while the second lemma bounds the change in the free-riding function for players who do not deviate.

Lemma 3. *Let $\alpha \geq 2$ let $s \in \mathcal{S}$ be a rational strategy profile such that $G(s)$ is a star. Let $K \subseteq [n]$ be a set of players and let $s' = (s'_K, s_{-K})$ be a profitable deviation for K, i.e., for all $i \in K$, it holds that $c_i(s'_K, s_{-K}) < c_i(s)$. Then for every $i \in K$ such that $deg_{G(s)}(i) = 1$ it holds that $f(s', i) > f(s, i)$ and $f(s', i) - f(s, i) \geq |s'_i| - |s_i| + 1$.*

Lemma 4. *Let $\alpha \geq 2$ and let $s \in \mathcal{S}$ be a rational strategy profile such that $G(s)$ is a star. Let $K \subseteq [n]$ be a player set and $s' = (s'_K, s_{-K})$ be a strategy profile that decreases the costs of all members of K. Then $\sum_{j \in [n] \setminus K} f(s', j) - f(s, j) > -|K|$. Moreover, if K contains a vertex i such that $deg_{G(s)}(i) > 1$ then $\sum_{j \in [n] \setminus K} f(s', j) - f(s, j) \geq 0$.*

Proof of Theorem 2. Let $s \in \mathcal{S}$ be a strategy profile that is rational such that $G(s)$ is a star. It is easy to see that s is a Nash equilibrium (see also [15]). Suppose that $K \in [n]$ and $s' \in \mathcal{S}_K$ are such that strategy profile $s' = (s'_K, s_{-K})$ decreases the costs of all players in K. Let $k = |K|$, we have two cases to consider.

If $\deg_{G(s)}(i) = 1$ for all $i \in K$, then $\sum_{i \in K}(f(s', i) - f(s, i)) \geq k + \sum_{i \in K}(|s'_i| - |s_i|) = k + \sum_{i \in [n]}(|s'_i| - |s_i|) = k + \sum_{i \in [n]}(f(s', i) - f(s, i))$, where the inequality follows from Lemma 3 and the last equality follows from (2). Hence $\sum_{i \in [n] \setminus K} f(s', i) - f(s, i) \leq -k$, which is the contradiction with Lemma 4.

If K contains the center vertex i (i.e., the vertex for which $\deg_{G(s)}(i) > 1$), then $\sum_{j \in K \setminus \{i\}}(f(s', j) - f(s, j)) \geq (k - 1) + \sum_{j \in K \setminus \{i\}}(|s'_j| - |s_j|) = (k - 1) + \sum_{j \in [n]}(|s'_j| - |s_j|) - (|s'_i| - |s_i|) = (k - 1) + \sum_{j \in [n]} (f(s', j) - f(s, j)) - (|s'_i| - |s_i|)$, where again the inequality follows from Lemma 3 and the last equality follows from (2).

Since i is a central vertex, we have $c^d_i(s') \geq c^d_i(s)$. Moreover, $i \in K$, hence $c_i(s') < c_i(s)$. This implies that $c^b_i(s') < c^b_i(s)$ or equivalently $|s'_i| < |s_i|$. So $\sum_{j \in K \setminus \{i\}}(f(s', j) -$

$f(s, j)) \geq k + \sum_{j\in[n]}(f(s', j) - f(s, j))$. Thus: $-k \geq \sum_{j\in[n]\setminus K}(f(s', j) - f(s, j)) + (f(s', i) - f(s, i)) \geq f(s', i) - f(s, i)$, where the last inequality follows from Lemma 4. On the other hand we have $f(s', i) - f(s, i) \geq -(k-1)$, since the change from s to s' could have removed at most $k-1$ edges going to player i, which is a contradiction. □

Next, for $\alpha > 2$, we present a family of strong equilibria none of which forms a star. The graphs resulting from these strong equilibria are trees of diameter 4.

Example 1. Our examples are paramatrized by two values $A \in \mathbb{N}$, $A \geq 4$ and $k \in \mathbb{N}$. Let $\alpha \geq 2n$, and let $n = Ak + 2$. In the following strategy profile s the only players who buy edges are $1, \ldots, A-1$ and n, i.e., for all $i \in [n]$, $A \leq i < n$, it holds that $s_i = \emptyset$. We denote player n by R.The total number of edges bought by players $\{1, \ldots, A-1, R\}$ is $n - 1 = Ak + 1$ such that $G(s)$ is a tree. $L_1 = \{A, A+1, \ldots, (A-1)k\}$ and $L_2 = \{(A-1)k + 1, \ldots, n-1\}$ denote the remaining $k+1$ players who do not buy edges. The strategy sets are defined as follows: Player R buys edges to L_2. Each player in $[A-1]$ buys an edge to $k-1$ players of L_1 in such a way that the degree in $G(s)$ equals 1 for every player in L_1. Moreover, each player in $[A-1]$ buys an edge to R. Thus, each player in $\{1, \ldots, A-1\}$ buys k edges, R buys $k+1$ edges, and all the remaining players (i.e., in L_1 and L_2) buy no edges and are leaves in $G(s)$. Figure 1 depicts this strategy profile.

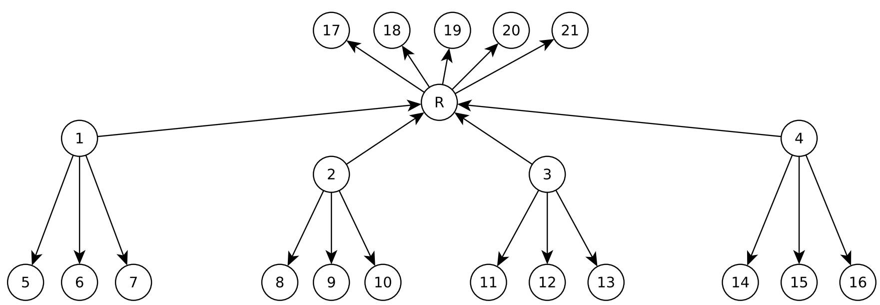

Fig. 1. Depiction of the graph $G(s)$ formed by the strong equilibrium s. The graph $G(s)$ is a is a tree of diameter 4. Strategy profile s is a strong equilibrium for $\alpha \geq 2n$ and $n = A \cdot k + 2$ where $A \in \mathbb{N}$, $A \geq 4$ is the number of players that buy edges and $k \in \mathbb{N}$. One player (called R) buys $k+1$ edges to leaves. The remaining $A-1$ players (that buy edges) each buy $k-1$ edges to leaves and one edge to R. In the depicted instance of the example we have: $A = 5$, $k = 4$, $L_1 = \{5, 6, \ldots, 16\}, L_2 = \{17, 18, \ldots, 21\}$ and the set of players buying edges is $\{1, \ldots, 4\} \cup \{R\}$.

Despite that s is relatively easy to define, establishing that s is a strong equilibrium is challenging.

Theorem 3. *If $\alpha \geq 2n$, strategy profile s forms a (non-star) tree and is a strong equilibrium.*

Proof. In s, there are four different types of node: The root R, the players $1 \ldots, A-1$, the leaves L_1, and the leaves L_2. The distance costs for each of these types are as follows.

$$c_i^d(s) = \begin{cases} 2n - A - k - 2 & \text{if } i = R & (3a) \\ 3n - A - 3k - 2 & \text{if } i \in [A-1] & (3b) \\ 3n - A - k - 4 & \text{if } i \in L_2 & (3c) \\ 4n - A - 3k - 4 & \text{if } i \in L_1 & (3d) \end{cases}$$

Proposition 2. *Let $s \in \mathcal{S}$. For all $i \in [n]$, $c_i^d(s) \geq= 2n - 2 - deg_{G(s)}(i)$.*

To show that s is a strong equilibrium, suppose for contradiction that $K \subseteq [n]$ and $s'_K \in \mathcal{S}_K$ are such that in $s' = (s'_K, s_{-K})$ it holds that $c_i(s') < c_i(s)$ for all $i \in K$. Under this assumption, using $(3a - 3d)$, we show that no player in K buys more edges under s' than it does under s.

Lemma 5. *For all $i \in K$, it holds that $|s'_i| \leq |s_i|$.*

The proofs of this lemma and the following lemma are omitted. Since $G(s)$ is a tree, it has the minimum number of edges among all connected graphs. Combining this with the lemma above yields that every player buys in s' *exactly* as many edges as in s.

Corollary 1. *Graph $G(s')$ is a tree, and for all $i \in [n]$, it holds that $|s'_i| = |s_i|$.*

Lemma 6. *Player R is not in K.*

Denote by $L_K = \{j \in L_1 \mid \exists i \in K : j \in s_i\}$ the leaves in L_1 that are directly connected to a player in K in $G(s)$. Let C_R be the players in the connected component of $G(\emptyset, s_{-K})$ containing R. Let $i \in \arg_{i'} \max\{d_{G(s')}(i', C_R) : i' \in K\}$ be a player in K that has the highest distance to C_R among all players in K.

Lemma 7. *The distance $d_{G(s')}(i, C_R)$ of i to C_R in $G(s')$ is as least* 2.

In s, the distance from i to C_R is 1. As C_R contains at least $k+2$ vertices, by deviating from s to s' the distance increase of player i to C_R is at least $k + 1$. We complete the proof of Theorem 3 by showing that by deviating from s to s', the distance decrease of player i to the players of $[n] \setminus C_R$ does not exceed $k + 1$. This is sufficient, as it implies that $c_i(s') \geq c_i(s)$ which contradicts that $i \in K$. To see this, observe that in $G(s)$ player i has in his neighborhood at most one player in K. If in $G(s')$ there are two or more players in K in i's neighborhood, then one of them is further away from C_R than i (contradicting the definition of i), or there is a cycle in $G(s')$ (contradicting Corollary 1). Let us separately compute the distance improvement to nodes in L_K and to nodes in K:

- In $G(s)$, the distance from i to all $|K| - 1$ players in K is 2. In $G(s')$ the distance from i to at most one player in K is 1, while at least $K - 2$ player are at distance 2 from i. Therefore, the total decrease in distance from i to players in K is at most 1.
- In $G(s)$, there are $k - 1$ players of L_K at distance 1 from i, and the remaining $|L_K| - k + 1$ players of L_K are at distance 3 from i. In $G(s')$ there are at most k players at distance 1 from i, there are at most $k - 1$ players at distance 2 from i (since the unique player i' of K that is directly connected to i (and buys the edge (i', i)) has at most $k - 1$ connections to L_K). Hence at least $|L_K| - 2k + 1$ players of L_K are at distance 3 from i. Therefore, the total decrease in distance from i to players in L_K is at most $(k - 1) + (3|L_K| - 3k + 3) - k - (2k - 2) - (3|L_K| - 6k + 3) = k + 1$.

It follows that by deviating from s to s', the maximum possible distance improvement for i to players in $[n] \setminus C_R$ is $k+2$, while the distance to at least $k+2$ vertices of C_R increases by 1. As $|s'_i| = |s_i|$ by Corollary 1, the building cost of i is not affected by the deviation, so the deviation is not profitable for i; a contradiction. □

5 Bounds on the Strong Price of Anarchy

In this section we analyze the strong price of anarchy of network creation games. First, for $\alpha < 2$, we provide exact expressions on the strong price of anarchy using the various insights of Section 4. Subsequently, for higher values of α, we provide a sequence of examples that converges to a price of anarchy of 3/2. This shows that the strong price of anarchy of the complete class of network creation games must lie in the interval $[3/2, 2]$, due to the upper bound of 2 established in [4]. It is trivial that for $\alpha \in (0, 1)$, the strong price of anarchy is 1. This holds because any rational strategy profile that forms the complete graph minimizes the social cost. The picture turns out to be relatively complex for $\alpha = 1$.

Theorem 4. *For $\alpha = 1$, the strong price of anarchy is $10/9$ if $n \in \{3, 4\}$, and the strong price of anarchy is $(3n+2)/3n$ if $n \geq 5$.*

Proof. By Theorem 1, for $\alpha = 1$ a strategy profile s is a strong equilibrium always if and only if it is rational and forms a graph of diameter at most 2 that is the complement of a forest. This means that vertices connected by an edge are distance 1 apart, and vertices not connected by an edge are distance 2 apart. A forest F has at most $n-1$ edges, so we obtain the following bound on the social cost of a strong equilibrium: $\alpha(n(n-1)/2-|F|)+2(2|F|+n(n-1)/2-|F|) = 3n(n-1)/2+|F| \leq 3n(n-1)/2+(n-1)$. This bound is achieved for $n \geq 5$ by taking for F any Hamiltonian path. Thus for $\alpha = 1$ and $n \geq 5$, given that the social optimum forms a complete graph, we obtain that the strong price of anarchy is $(3n(n-1)/2+(n-1))/(3n(n-1)/2) = (3n(n-1)+2(n-1))(3n(n-1)) = 3n+2/3n$. For $n = 4$, the maximum size forest (such that the complement of it has diameter 2) has only 2 edges, and for $n = 3$ it has only 1 edge. Therefore, the strong price of anarchy for $\alpha = 1$ and $n \in \{3, 4\}$ equals 10/9. □

For $\alpha \in (1, 2)$, there exists no strong equilibrium if $n \geq 5$ (see [4]). Therefore, it remains to derive the strong equilibria for $\alpha \in (1, 2)$ and $n \in \{3, 4\}$.

Proposition 3. *For $\alpha \in (1, 2)$ the strong price of anarchy is $(2\alpha+8)/(3\alpha+6)$ if $n = 3$, and the strong price of anarchy is $(4\alpha+16)/(6\alpha+12)$ if $n = 4$.*

For $\alpha > 2$ it seems very challenging to prove precise bounds on the strong price of anarchy. However, it is known that for $\alpha \geq 2$ the strong price of anarchy is at most 2 [4]. We now complement this bound by showing that for Example 1 (given in Section 4), the strong price of anarchy is at least 3/2.

Theorem 5. *The price of anarchy of network creation games is at least 3/2.*

Proof. Let $x \geq 4$ and consider the strong equilibrium s given in Example 1, for $\alpha = 2n$ and $k = A = x$. The players in L_1 each have a distance cost of $4n - 4 - A - 3k = 4x^2 + 4 - x - 3x$. Since $|L_1| = (A-1)(k-1) = x^2 - 2x + 1$ the total distance cost of s is at least $4x^4 - 12x^3 + 16x^2 - 12x + 4$. Moreover, $G(s)$ is a tree, so the total building cost of s equals $(n-1)\alpha = (Ak+1)2(Ak+2) = 2x^4 + 6x^2 + 4$. Therefore, the social cost of s satisfies $C(s) \geq 6x^4 - 12x^3 + 22x^2 - 12x + 4$.

For $\alpha \geq 2$, the social optimum forms an n-star. Thus, the optimal social cost is $(n-1)\alpha + 2(n-1)^2 = 2n(n-1) + 2(n-1)^2 \leq 4n(n-1) = 4x^4 + 12x^2 + 8$ Combining these two bounds and taking x to infinity, we obtain that the strong price of anarchy is at least $\lim_{x\to\infty}(6x^4 - 12x^3 + 22x^2 - 12x + 4)/(4x^4 + 12x^2 + 8) = 3/2$. □

6 Convergence of Coalitional Improvement Dynamics

In this section we study the c-FIP and coalitional weak acyclicity of network creation games. On the positive side, c-weak acyclicity holds for $\alpha \in (0, 2)$[3] and for all $\alpha \leq n/2$ in case the starting strategy profile forms a tree. On the other hand, our negative results encompass that the c-FIP is not satisfied for any α.[4] First, running best response dynamics on a network creation game ends up in a pure Nash equilibrium.

Lemma 8. *For $\alpha < 1$, every network creation game has the FBRP.*

From Lemma 8 and the fact that Nash equilibria and strong equilibria coincide for $\alpha < 1$ (as we also pointed out in Section 4), we obtain as a corollary that for $\alpha < 1$, every network creation game is c-weakly acyclic. For $\alpha = 1$ we can also show weak acyclicity and c-weak acyclicity.

Lemma 9. *For $\alpha = 1$, every network creation game is weakly acyclic and c-weakly acyclic.*

We may also prove that for $\alpha \in (1, 2)$ and $n \in \{3, 4\}$, network creation games are c-weakly acyclic. (Recall that for $\alpha \in (1, 2)$ and $n \geq 5$, strong equilibria do not exist.)

Proposition 4. *For $\alpha > 1$ and $n = 3$ network creation games have the c-FIP. For $\alpha \in (1, 2)$ and $n = 4$, network creation games are c-weakly acyclic.*

For $\alpha \leq n/2$ we can show that c-weak acyclicity is satisfied as long as our starting strategy profile forms a tree. This result relies on the following lemma about centroid vertices of trees.

Lemma 10. *Let $T = (V, E)$ be a tree, and let $v \in V$ be a centroid vertex of T. It holds that* $\max\{|V_i| : (V_i, E_i) \in C_{T-v}\} \leq (1/2)|V|$.

Lemma 11. *For $\alpha \in (1, n/2)$, let $s \in \mathcal{S}$ be such that $G(s)$ is a tree. Then there exists an improvement path resulting in a strong equilibrium. Hence, every network creation game is weakly acyclic and c-weakly acyclic with respect to trees.*

[3] Except for $\alpha \in (1, 2)$ and $n \geq 5$, in which case we know that strong equilibria do not exist.

[4] An exception to this is that we can prove that the coalitional finite improvement property is satisfied for the very special case that $\alpha > 1$ and $n = 3$.

Proof. Let $s \in \mathcal{S}$ and suppose $G(s)$ is a tree. Let $v \in [n]$ be a centroid vertex of $G(s)$. Consider the following sequence of deviations. If there is a player i such that $d_{G(s)}(i, v) \geq 2$, then $s'_i = s_i \cup \{v\}$ and $s' = (s'_i, s_{-i})$. Repeat this step with $s = s'$ until $d_{G(s)}(i, v) = 1$ for all $i \in V \setminus \{v\}$. Observe that since v is a centroid vertex of $G(s)$, by Lemma 10, player i decreases the distance to at least $n/2$ players by at least 1 by buying an edge to v. This exceeds the cost of α, hence this deviation is profitable. Otherwise, if there is no player i such that $d_{G(s)}(i, v) \geq 2$, and $G(s)$ is not a star, then there are players $i, j \in [n]$ such that $i \neq v$, $j \neq v$ and $j \in s_i$, then let $s'_i = s_i \setminus \{j\}$. Repeat this step until $G(s)$ is a star. Observe that player i is better off by the strategy change. She saves $\alpha > 1$ in her building cost and her distance cost increases by only 1, since for each player not in i's neighborhood there is a shortest path through v. Hence the only loss is the distance increase between i and j. If s is rational after this sequence of deviations, then we have reached a strong equilibrium by Theorem 2. Otherwise there are i, j such that $i \in s_j$ and $j \in s_i$. We set $s'_i = s_i \setminus \{j\}$ and repeat this step until we reach a rational s. □

However, we may show that in general, network creation games do not have the c-FIP, regardless of the choice of α.

Theorem 6. *For every α there exists a number of players n such that network creation game (n, α) does not have the c-FIP.*

This above theorem is proved by providing examples for $\alpha < 1, \alpha = 1, \alpha \in (1, 2)$, and $\alpha = 2$ separately. For $\alpha > 2$, the example in Theorem 1 of [9] implies that network creation games are not potential games. Hence they do not possess the FIP and the c-FIP for this range of α.

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