On the Inefficiency of Standard Multi-Unit Auctions

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Abstract

We study two standard multi-unit auction formats for allocating multiple units of a single good to multi-demand bidders. The first one is the Discriminatory Auction, which charges every winner his winning bids. The second is the Uniform Price Auction, which determines a uniform price to be paid per unit. Variants of both formats find applications ranging from the allocation of state bonds to investors, to online sales over the internet, facilitated by popular online brokers.

For these formats, we consider two bidding interfaces: (i) standard bidding, which is most prevalent in the scientific literature, and (ii) uniform bidding, which is more popular in practice. In this work, we evaluate the economic inefficiency of both multi-unit auction formats for both bidding interfaces, by means of upper and lower bounds on the Price of Anarchy for pure Nash equilibria and mixed Bayes-Nash equilibria. Our developments improve significantly upon bounds that have been obtained recently in [Markakis, Telelis, SAGT 2012] and [Syrgkanis, Tardos, STOC 2013] for submodular valuation functions. Moreover, we consider for the first time bidders with subadditive valuation functions for these auction formats. Our results signify that these auctions are nearly efficient, which provides further justification for their use in practice.

1 Introduction

We study standard multi-unit auction formats for allocating multiple units of a single good to multi-demand bidders. Multi-unit auctions are one of the most widespread and popular tools for selling identical units of a good with a single auction process. In practice, they have been in use for a long time, one of their most prominent applications being the auctions offered by the U.S. and U.K. Treasuries for selling bonds to investors, see e.g., the U.S. treasury report [22]. In more recent years, they are also implemented by various online brokers [6, 17]. In the literature, multi-unit auctions have been a subject of study ever since the seminal work of Vickrey [23] (although the need for such a market enabler was conceived even earlier, by Friedman, in [9]) and the success of these formats has led to a resurgence of interest in auction design.

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Valuation Functions	**Auction Format** (`bidding: standard` | `uniform`) *Discriminatory Auction*	*Uniform Price Auction*
Submodular	$\frac{e}{(e-1)}$	3.1462
Subadditive	$2 \mid \frac{2e}{(e-1)}$	$4 \mid 6.2924$

Table 1: Upper bounds on the (Bayesian) economic inefficiency of multi-unit auctions.

There are three simple *Standard Multi-Unit Auction* formats that have prevailed and are being implemented; these are the *Discriminatory Auction*, the *Uniform Price Auction* and the *Vickrey Multi-Unit Auction*. All three formats share a common allocation rule and bidding interface and have seen extensive study in auction theory [12, 15]. Each bidder under these formats is asked to issue a sequence of non-increasing marginal bids, one for each additional unit. For an auction of k units, the k highest marginal bids win, and each grants its issuing bidder a single unit. The formats differ in the way that payments are determined for the winning bidders. The Discriminatory Auction prescribes that each bidder pays the sum of his winning bids. The Uniform Price Auction charges the lowest winning or highest losing marginal bid per allocated unit. The Vickrey auction charges according to an instance of the Clarke payment rule (thus being a generalization of the well known single-item Second-Price auction).

Except for the Vickrey auction, which is truthful and efficient, the others suffer from a *demand reduction* effect [1], whereby bidders may have incentives to understate their value, so as to receive less units at a better price. This effect is amplified when bidders have non-submodular valuation functions, since the bidding interface forces them to encode their value within a submodular bid vector. Even worse, in many practical occasions bidders are asked for a *uniform bid* per unit together with an upper bound on the number of desired units. In such a setting, each bidder is required to "compress" his valuation function into a bid that scales linearly with the number of units. The mentioned allocation and pricing rules apply also in this *uniform bidding* setting, thus yielding different versions of Discriminatory and Uniform Price Auctions. Despite the volume of research from the economics community [1, 16, 7, 3, 18, 4] and the widespread popularity of these auction formats, the first attempts of quantifying their economic efficiency are only very recent [14, 20]. There has also been no study of these auction formats for non-submodular valuations, as noted by Milgrom [15].

Our Contributions. We study the inefficiency of the Discriminatory Auction and Uniform Price Auction under the standard and uniform bidding interfaces. Our main results are improved inefficiency bounds for bidders with submodular valuation functions and new bounds for bidders with subadditive valuation functions.[1] The results are summarized in Table 1.

[1] To the best of our knowledge, for subadditive valuation functions our bounds provide the first quantification of the inefficiency of these auction formats.

Our bounds indicate that these auctions are nearly efficient, which paired with their simplicity provides further justification for their use in practice.

Our focus is mostly on the efficiency of Bayes-Nash equilibria (see Section 3), but we also discuss pure Nash equilibria (see **Appendix A**). For submodular valuation functions, we derive upper bounds of $\frac{e}{e-1}$ and $3.1462 < \frac{2e}{e-1}$ for the Discriminatory and the Uniform Price Auctions, respectively. These improve upon the previously best known bounds of $\frac{2e}{e-1}$ and $\frac{4e}{e-1}$ [20]. For the Uniform Price Auction, our bound is less than a factor 2 away from the known lower bound of $\frac{e}{e-1}$ [14]. We also prove lower bounds of $\frac{e}{e-1}$ and 2 for the Discriminatory Auction and Uniform Price Auction, with respect to the currently known proof techniques [20, 8, 5, 2, 11]. As a consequence, unless the upper bound of $\frac{e}{e-1}$ for the Discriminatory Auction is tight, its improvement requires the development of novel tools; the same holds for reducing the Uniform Price Auction upper bound below 2 (if $\frac{e}{e-1}$ from [14] is indeed worst-case). For subadditive valuations, we obtain bounds of $\frac{2e}{e-1}$ and $6.2924 < \frac{4e}{e-1}$ for Discriminatory and Uniform Price Auctions respectively, independent of the bidding interface. Further, for the standard bidding interface we are able to derive improved bounds of 2 and 4, respectively, by adapting a recent technique from [8]. We also give a lower bound of almost 2 for uniform pricing and subadditive valuations. In Section 4 we discuss further applications of our results in connection with the smoothness framework of [20]. In particular, some of our bounds carry over to simultaneous and sequential compositions of such auctions (see Table 2 in Section 4).

Related Work. The multi-unit auction formats that we examine here present technical and conceptual resemblance to the *Simultaneous Auctions* format that has received significant attention recently [8, 5, 2, 11, 20]. However, upper bounds in this setting do not carry over to our format. Simultaneous auctions were first studied by Christodoulou, Kovacs and Schapira [5]. The authors proposed that each of a collection of distinct goods, with one unit available for each of them, is sold in a distinct *Second Price Auction*, simultaneously and independently of the other goods. Bidders in this setting may have combinatorial valuation functions over the subsets of goods, but they are forced to bid separately for each good. For bidders with fractionally subadditive valuation functions, they proved a tight upper bound of 2 on the mixed Bayesian Price of Anarchy of the Simultaneous Second Price Auction. Bhawalkar and Roughgarden [2] extended the study of inefficiency for subadditive bidders and showed an upper bound of $O(\log m)$ which was recently reduced to 4 by Feldman *et al.* [8]. For arbitrary valuation functions, Fu, Kleinberg and Lavi [10] proved an upper bound of 2 on the inefficiency of pure Nash equilibria, when they exist.

Hassidim *et al.* [11] studied *Simultaneous First Price Auctions*. They showed that pure Nash equilibria in this format are always efficient, when they exist. They proved constant upper bounds on the inefficiency of mixed Nash equilibria for (fractionally) subadditive valuation functions and $O(\log m)$ and $O(m)$ for the inefficiency of mixed Bayes-Nash equilibria for subadditive and arbitrary valuation functions. Syrgkanis showed in [21] that this format has Bayesian Price of Anarchy $\frac{e}{e-1}$ for fractionally subadditive valuation functions. Feldman *et al.* [8] proved an upper bound of 2 for subadditive ones.

Recently, Syrgkanis and Tardos [20] and Roughgarden [19] independently developed extensions of the *smoothness technique* for games of incomplete information. In [20], these ideas are further developed for analyzing the inefficiency of simultaneous and sequential *compositions* of simple auction mechanisms. They demonstrate extensive applications of their techniques on welfare analysis of standard multi-unit auction formats *and* their compositions. For submodular valuation functions, they prove inefficiency upper bounds of $\frac{2e}{e-1}$ and $\frac{4e}{e-1}$ for the Discriminatory Auction and Uniform Price Auction, respectively. Here, we improve upon these results; our improvements carry over to simultaneous and sequential compositions as well.

2 Definitions and Preliminaries

We consider auctioning k units of a single good to a set $[n] = \{1, ..., n\}$ of n bidders, indexed by $i = 1, \dots, n$. Every bidder $i \in [n]$ has a non-negative non-decreasing private valuation function $v_i : (\{0\} \cup [k]) \mapsto \Re^+$ over quantities of units, where $v_i(0) = 0$. We denote by $\mathbf{v} = (v_1, \dots, v_n)$ the *valuation function profile* of bidders. We consider in particular (symmetric) *submodular* and *subadditive* functions:

Definition 1 *A valuation function $f : (\{0\} \cup [k]) \mapsto \Re^+$ is called:*
- *submodular iff for every $x < y$, $f(x) - f(x-1) \geq f(y) - f(y-1)$.*
- *subadditive iff for every x, y, $f(x+y) \leq f(x) + f(y)$.*

The class of submodular functions is strictly contained in the class of subadditive ones [13]. For any non-negative non-decreasing function $f : (\{0\} \cup [k]) \mapsto \Re^+$ and any integers $x, y \in [k], x < y$, the following are known to hold: If f is submodular, then $f(x)/x \geq f(y)/y$. If f is subadditive, then $f(x)/x \geq f(y)/(x+y)$.

A valuation function v_i can be specified by a vector $\mathbf{m}_i = (m_i(1), ..., m_i(k))$ of the *marginal values* $m_i(j) = v_i(j) - v_i(j-1)$ of bidder i, for each additional unit in his allocation (if v_i is submodular, $m_i(j) \geq m_i(j+1)$).

Standard multi-unit auctions. The *standard format*, as described in auction theory [12, 15], prescribes that each bidder $i \in [n]$ submits a vector of k non-negative non-increasing *marginal bids* $\mathbf{b}_i = (b_i(1), \dots, b_i(k))$ with $b_i(1) \geq \dots \geq b_i(k)$. We will often refer to these simply as *bids*. In the *uniform bidding format*, each bidder i submits only a single bid $\bar{b}_i$ along with a quantity $q_i \leq k$, the interpretation being that i is willing to pay at most $\bar{b}_i$ per unit for up to q_i units.

The allocation rule of standard multi-unit auctions grants the issuer of each of the k highest (marginal) bids a distinct unit per winning bid. The pricing rule differentiates the formats. Let $x_i(\mathbf{b})$ be the number of units won by bidder i under profile $\mathbf{b} = (\mathbf{b}_1, \dots, \mathbf{b}_n)$. We study the following two pricing rules:

(i) *Discriminatory Pricing.* Every bidder i pays for every unit a price equal to his corresponding winning bid, i.e., the utility of i is
$$u_i^{v_i}(\mathbf{b}) = v_i\left(x_i(\mathbf{b})\right) - \sum_{j=1}^{x_i(\mathbf{b})} b_i(j).$$

(ii) *Uniform Pricing.* Every bidder i pays for every unit a price equal to the *highest losing bid* $p(\mathbf{b})$, i.e., the utility of i is
$$u_i^{v_i}(\mathbf{b}) = v_i\left(x_i(\mathbf{b})\right) - x_i(\mathbf{b})p(\mathbf{b}).$$

For a bidding profile $\mathbf{b}$, the produced allocation $\mathbf{x}(\mathbf{b}) = (x_1(\mathbf{b}), x_2(\mathbf{b}), \dots, x_n(\mathbf{b}))$ has a *social welfare* equal to the bidders' total value: $SW(\mathbf{v}, \mathbf{b}) = \sum_{i=1}^{n} v_i(x_i(\mathbf{b}))$. The (pure) Price of Anarchy is the worst case ratio, over all pure Nash equilibrium profiles $\mathbf{b}$, of the optimal social welfare over $SW(\mathbf{v}, \mathbf{b})$.

Incomplete Information. Under the incomplete information model of Harsanyi, the valuation function $\mathbf{v}_i$ of bidder i is drawn from a finite set V_i according to a discrete probability distribution $\pi_i : V_i \to [0, 1]$ (independently of the other bidders); we will write

$\mathbf{v}_i \sim \pi_i$. The actual drawn valuation function of every bidder is *private*. A valuation profile $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathcal{V} = \times_{i\in[n]} V_i$ is drawn from a *publicly known distribution* $\pi : \mathcal{V} \to [0,1]$, where π is the product distribution of $\pi_1, \dots, \pi_n$, i.e., $\pi(\mathbf{v}) \mapsto \prod_{i\in[n]} \pi_i(\mathbf{v}_i)$. Every bidder i knows his own valuation function but does not know the valuation function $\mathbf{v}_{i'}$ drawn by any other bidder $i' \neq i$. Bidder i may only use his knowledge of π to estimate $\mathbf{v}_{-i}$. Given the publicly known distribution π, the (possibly mixed) strategy of every bidder is a function of his own valuation $\mathbf{v}_i$, denoted by $B_i(\mathbf{v}_i)$. B_i maps a valuation function $\mathbf{v}_i \in V_i$ to a *distribution* $B_i(\mathbf{v}_i) = B_i^{v_i}$, over all possible bid vectors for i. In this case we will write $\mathbf{b}_i \sim B_i^{v_i}$, for any particular bid vector $\mathbf{b}_i$ drawn from this distribution. We also use the notation $\mathbf{B}_{-i}^{\mathbf{v}_{-i}}$, to refer to the vector of randomized strategies of bidders other than i, under profile $\mathbf{v}_{-i}$.

A *Bayes-Nash equilibrium* (BNE) is a strategy profile $\mathbf{B} = (B_1, \dots, B_n)$ such that for every bidder i and for every valuation $\mathbf{v}_i$, $B_i(\mathbf{v}_i)$ maximizes the utility of i in expectation, over the distribution of the other bidders' valuations $\mathbf{w}_{-i}$ *given* $\mathbf{v}_i$ and over the distribution of i's own and the other bidders' strategies, $\mathbf{B}^{(\mathbf{v}_i, \mathbf{w}_{-i})}$, i.e., for every pure strategy $\mathbf{c}_i$ of i:

$$\mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i,\ \mathbf{b}\sim\mathbf{B}^{(\mathbf{v}_i,\mathbf{w}_{-i})}}\left[u_i^{v_i}(\mathbf{b})\right] \geq \mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i,\ \mathbf{b}_{-i}\sim\mathbf{B}^{\mathbf{w}_{-i}}}\left[u_i^{v_i}(\mathbf{c}_i, \mathbf{b}_{-i})\right]$$

where $\mathbb{E}_{\mathbf{v}}$ and $\mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i}$ denote the expectation over the distributions π and $\pi(\cdot|\mathbf{v}_i)$ (i.e., given $\mathbf{v}_i$), respectively.

Fix a valuation profile $\mathbf{v} \in \mathcal{V}$ and consider a (mixed) bidding configuration $\mathbf{B}^{\mathbf{v}}$ under $\mathbf{v}$. The Social Welfare $SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}})$ under $\mathbf{B}^{\mathbf{v}}$ when the valuations are $\mathbf{v}$ is defined as the expectation over the bidding profiles chosen by the bidders from their randomized strategies, i.e., $SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}}) = \mathbb{E}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{v}}}\left[\sum_i v_i(x_i(\mathbf{b}))\right]$. The *expected* Social Welfare in Bayes-Nash equilibrium $\mathbf{B}^{\mathbf{v}}$ is then $\mathbb{E}_{\mathbf{v}\sim\pi}[SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}})]$. The socially optimum assignment under valuation profile $\mathbf{v} \in \mathcal{V}$ will be denoted by $\mathbf{x}^{\mathbf{v}}$. The *expected* optimum social welfare is then $\mathbb{E}_{\mathbf{v}}[SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}})]$. Under these definitions, we will study the *Bayesian Price of Anarchy*, i.e., the worst case ratio $\mathbb{E}_{\mathbf{v}}[SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}})]/\mathbb{E}_{\mathbf{v}}[SW(\mathbf{v}, \mathbf{B}^{\mathbf{v}})]$ over all possible product distributions π and Bayes-Nash equilibria $\mathbf{B}$ for π.[2]

Similarly to previous works, when analyzing the Uniform Price Auction we assume *no-overbidding*, i.e., each bidder never bids more than his value for every number of units; formally, for every $s \in [k]$, $\sum_{j\in[s]} b_i(j) \leq v_i(s)$. In our analysis, we will use $\beta_j(\mathbf{b})$ to refer to the j-th lowest winning bid under profile $\mathbf{b}$; thus $\beta_1(\mathbf{b}) \leq \cdots \leq \beta_k(\mathbf{b})$.

3 Bayes-Nash Inefficiency

Our main results concern the inefficiency of Bayes-Nash equilibria. For a discussion on the properties of pure Nash equilibria, we refer the reader to **Appendix A**. We derive bounds on the (mixed) Bayesian Price of Anarchy for the Discriminatory and the Uniform Price Auctions with submodular and subadditive valuation functions. For the latter class our bounds are the first results to appear in the literature of standard multi-unit auctions (see also the commentary in [15, Chapter 7]). Some proofs of this section are deferred to **Appendix B**.

Theorem 1 *The Bayesian Price of Anarchy (under the standard or uniform bidding format) is at most*

[2] As in previous works [5, 8], we ensure existence of Bayes-Nash equilibria in our auction formats by assuming that bidders have bounded and finite strategy spaces, e.g., derived through discretization. Our bounds on the auctions' Bayesian inefficiency hold for sufficiently fine discretizations (see also Appendix D of [8]).

(i) $\frac{e}{e-1}$ *and* $\frac{2e}{e-1}$ *for the Discriminatory Auction with submodular and subadditive valuation functions, respectively,*

(ii) $|W_{-1}(-1/e^2)| \approx 3.1462 < \frac{2e}{e-1}$ *and* $2|W_{-1}(-1/e^2)| \approx 6.2924 < \frac{4e}{e-1}$ *for the Uniform Price Auction with submodular and subadditive valuation functions, respectively,* W_{-1} *being the lower branch of the Lambert* W *function.*

This theorem improves on the currently best known upper bounds of $\frac{2e}{e-1}$ and $\frac{4e}{e-1}$ for the Discriminatory Auction and the Uniform Price Auction, respectively, with submodular valuation functions due to Syrgkanis and Tardos [20]. For the Uniform Price Auction, this further reduces the gap from the known lower bound of $\frac{e}{e-1}$ [14]. Syrgkanis and Tardos [20] obtained their bounds through an adaptation of the *smoothness framework* for games with incomplete information ([19, 21]). The bounds of Theorem 1 and some additional results can also be obtained through this framework. We comment on this in more detail in Section 4.

For subadditive valuation functions and the standard bidding format, however, better bounds can be obtained by adapting a technique recently introduced by Feldman *et al.* [8], which does not fall within the smoothness framework. We were unable to derive these bounds via a smoothness argument and believe that this is due to the additional flexibility provided by this technique.

Theorem 2 *The Bayesian Price of Anarchy is at most* 2 *and* 4 *for the Discriminatory Auction and the Uniform Price Auction, respectively, with subadditive valuation functions under the standard bidding format.*

3.1 Proof Template for Bayesian Price of Anarchy

In order to present all our bounds from Theorem 1 and Theorem 2 in a self-contained and unified manner, we make use of a proof template which is formalized in Theorem 3 below. Variants of this approach have been used in several previous works (e.g., [14, 5, 2]).

Theorem 3 *Let* V *be a class of valuation functions. Suppose that for every valuation profile* $\mathbf{v} \in V^n$*, for every bidder* $i \in [n]$*, and for every distribution* $\mathcal{P}_{-i}$ *over non-overbidding profiles* $\mathbf{b}_{-i}$*, there is a bidding profile* $\mathbf{b}'_i$ *such that the following inequality holds for some* $\lambda > 0$ *and* $\mu \geq 0$*:*

$$\mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}_{-i}}\left[u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i})\right] \geq \lambda \cdot v_i(x_i^{\mathbf{v}}) - \mu \cdot \mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}_{-i}}\left[\sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i})\right]. \quad (1)$$

Then the Bayesian Price of Anarchy is at most

(i) $\max\{1, \mu\}/\lambda$ *for the Discriminatory Auction,*

(ii) $(\mu + 1)/\lambda$ *for the Uniform Price Auction.*

Note that in this theorem we make no assumptions regarding the bidding interface; proving a bound for the uniform bidding interface only requires that we exhibit a uniform bidding strategy $\mathbf{b}'_i$ for each bidder i and for any distribution $\mathcal{P}_{-i}$ of uniform non-overbidding profiles $\mathbf{b}_{-i}$.

In Section 3.3 we show that our bound of $\frac{e}{e-1}$ for the Discriminatory Auction is essentially best possible if one sticks to the proof template of Theorem 3; which also rules out that better bounds can be obtained via the recent techniques in [20, 8].

3.2 Key Lemma and Proofs of Theorem 1 and Theorem 2

The following is our key lemma to prove Theorem 1. We point out that it applies to arbitrary valuation functions and to any multi-unit auction which is *discriminatory price dominated*, i.e., the total payment $P_i(\mathbf{b})$ of bidder i under profile $\mathbf{b}$ satisfies $P_i(\mathbf{b}) \leq \sum_{j\in[x_i(\mathbf{b})]} b_i(j)$. Note that every multi-unit auction guaranteeing *individual rationality* must satisfy this condition.

Lemma 1 (Key Lemma) *Let $\mathbf{v}$ be a valuation profile and suppose that the pricing rule is discriminatory price dominated. Define $\tau_i = \arg\min_{j\in[x_i^{\mathbf{v}}]} v_i(j)/j$ for every $i \in [n]$. Then for every bidder $i \in [n]$ and every bidding profile $\mathbf{b}_{-i}$ there exists a randomized uniform bidding profile $\mathbf{b}_i'$ such that for every $\alpha > 0$*

$$\mathbb{E}[u_i^{\mathbf{v}_i}(\mathbf{b}_i', \mathbf{b}_{-i})] \geq \alpha \left(1 - \frac{1}{e^{1/\alpha}}\right) x_i^{\mathbf{v}} \frac{v_i(\tau_i)}{\tau_i} - \alpha \sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}). \tag{2}$$

Proof. Define $B = (1-e^{-1/\alpha})$ and let $\mathbf{c}_i$ be the vector that is $v_i(\tau_i)/\tau_i$ on the first $x_i^{\mathbf{v}}$ entries, and is 0 everywhere else. Let t be a random variable drawn from $[0, B]$ with probability density function $f(t) = \alpha/(1-t)$. Define the random deviation of bidder i as $\mathbf{b}_i' = t\mathbf{c}_i$. Note that $\mathbf{b}_i'$ is always a uniform bid vector.

Let k^* be the number of items that bidder i would win under profile $(B\mathbf{c}_i, \mathbf{b}_{-i})$, i.e., the number of items won by i, when i would deviate to bid vector $B\mathbf{c}_i$. For $j = 0, \dots, k^*$, let γ_j refer to the infimum value in $[0, B]$ such that bidder i would win j items if he would deviate to bid vector $\gamma_j \mathbf{c}_i$. Note that this definition is equivalent to defining γ_j as the least value in $[0, B]$ that satisfies $\gamma_j v_i(\tau_i)/\tau_i = \beta_j(\mathbf{b}_{-i})$. For notational convenience, we define $\gamma_{k^*+1} = B$.

Let $x_i(\mathbf{b}_i', \mathbf{b}_{-i})$ be the random variable that denotes the number of units allocated to bidder i under $(\mathbf{b}_i', \mathbf{b}_{-i})$. It always holds that $x_i(\mathbf{b}_i', \mathbf{b}_{-i}) \leq k^* \leq x_i^{\mathbf{v}}$, because bidder i bids $b_i'(j) = 0$ for all $j = x_i^{\mathbf{v}} + 1, \dots, k$. More precisely, we have $x_i(\mathbf{b}_i', \mathbf{b}_{-i}) = j$ if $t \in (\gamma_j, \gamma_{j+1}]$ for $j = 0, \dots, k^*$. By assumption, the payment of bidder i under profile $(\mathbf{b}_i', \mathbf{b}_{-i})$ is at most $t x_i(\mathbf{b}_i', \mathbf{b}_{-i}) v_i(\tau_i)/\tau_i$. Also note that, by definition of τ_i, it holds that $v_i(j) \geq j v_i(\tau_i)/\tau_i$ for $j \leq x_i^{\mathbf{v}}$. Using these two facts, we can bound the expected utility of bidder i as follows:

$$\begin{aligned}
\mathbb{E}[u_i^{\mathbf{v}_i}(\mathbf{b}_i', \mathbf{b}_{-i})] &\geq \sum_{j=1}^{k^*} \int_{\gamma_j}^{\gamma_{j+1}} \left(v_i(j) - tj\frac{v_i(\tau_i)}{\tau_i}\right) f(t)dt \\
&\geq \sum_{j=1}^{k^*} \int_{\gamma_j}^{\gamma_{j+1}} j\frac{v_i(\tau_i)}{\tau_i}(1-t)f(t)dt = \alpha \sum_{j=1}^{k^*} j\frac{v_i(\tau_i)}{\tau_i} \int_{\gamma_j}^{\gamma_{j+1}} 1dt \\
&= \alpha \sum_{j=1}^{k^*} j\frac{v_i(\tau_i)}{\tau_i}(\gamma_{j+1} - \gamma_j) = \alpha B k^* \frac{v_i(\tau_i)}{\tau_i} - \alpha \sum_{j=1}^{k^*} \gamma_j \frac{v_i(\tau_i)}{\tau_i} \\
&= \alpha B k^* \frac{v_i(\tau_i)}{\tau_i} - \alpha \sum_{j=1}^{k^*} \beta_j(\mathbf{b}_{-i}) \geq \alpha B x_i^{\mathbf{v}} \frac{v_i(\tau_i)}{\tau_i} - \alpha \sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}).
\end{aligned}$$

The last inequality holds because $Bv_i(\tau_i)/\tau_i \leq \beta_j(\mathbf{b}_{-i})$, for $k^*+1 \leq j \leq x_i^{\mathbf{v}}$, by the definition of k^*. The above derivation implies (2). ☐

The deviation $\mathbf{b}_i'$ defined in Lemma 1 is a distribution on uniform bidding strategies. That is, the lemma applies to both the standard and the uniform bidding format. Observe also that $\mathbf{b}_i'$ satisfies the no-overbidding assumption.

Proof. [of Theorem 1] First consider the case of submodular valuation functions. In this case, $\tau_i = x_i^{\mathbf{v}}$ for every $i \in [n]$, as explained in Section 2. Using our Key Lemma, we conclude that Theorem 3 holds for $(\lambda, \mu) = (\alpha\left(1 - e^{-1/\alpha}\right), \alpha)$. The stated bounds are obtained by choosing $\alpha = 1$ for the Discriminatory Auction and $\alpha = -1/(W_{-1}(-1/e^2) + 2) \approx 0.87$ for the Uniform Price Auction.

Next consider the case of subadditive valuation functions. The following lemma shows that subadditive valuation functions can be approximated by uniform ones, thereby losing at most a factor 2.

Lemma 2 *If v_i is subadditive, then $\frac{v_i(\tau_i)}{\tau_i} \geq \frac{1}{2}\frac{v_i(x_i^{\mathbf{v}})}{x_i^{\mathbf{v}}}$ with $\tau_i = \arg\min_{j \in [x_i^{\mathbf{v}}]} \frac{v_i(j)}{j}$.*

By combining Lemma 2 with our Key Lemma, it follows that Theorem 3 holds for $(\lambda, \mu) = (\frac{\alpha}{2}\left(1 - e^{-1/\alpha}\right), \alpha)$. The bounds stated in Theorem 1 are obtained by the same choices of α as for the submodular valuation functions. □

Next, consider subadditive valuations under the *standard* bidding format. We derive improved bounds of 2 and 4 for the Discriminatory and Uniform Price Auction, respectively. To this end, we adapt an approach recently developed by Feldman *et al.* [8] to establish an analog of our Key Lemma. The main idea is to construct the bid $\mathbf{b}_i'$ by using the distribution $\mathcal{P}_{-i}$ on the profiles $\mathbf{b}_{-i}$. Theorem 2 then follows from Theorem 3 in combination with Lemma 3 below.

Lemma 3 *Let V be the class of subadditive valuation functions. Then Theorem 3 holds true with $(\lambda, \mu) = (\frac{1}{2}, 1)$ for the Discriminatory and $(\lambda, \mu) = (\frac{1}{2}, 1)$ for the Uniform Price Auction (under the standard bidding format).*

3.3 Lower Bounds

A lower bound of approximately $\frac{e}{e-1}$ for Uniform Price Auctions with submodular bidders was given in [14]. Thus, our upper bound for this case is less than a factor 2 away. For subadditive valuation functions, we prove in **Appendix B** a lower bound of almost 2

Theorem 4 *The Price of Anarchy is at least $\frac{2k}{k+1}$ for the Uniform Price Auction with subadditive valuations (under the uniform bidding interface).*

No lower bound is known for the Discriminatory Auction, although *Demand Reduction* (which is responsible for welfare loss in this format) has been observed previously [12, 1]. In light of this, we prove here an *impossibility result* showing that for the Discriminatory Auction no bound better than $\frac{e}{e-1}$ on the Price of Anarchy can be achieved via the proof template given in Theorem 3. Similarly, for the Uniform Price Auction we rule out that a bound better than 2 on the Price of Anarchy can be derived through this template.

Theorem 5 *There is a lower bound of $\frac{e}{e-1}$ and 2 on the Bayesian Price of Anarchy for the Discriminatory Auction and the Uniform Price Auction, respectively, with submodular valuation functions that can be derived through the proof template given in Theorem 3.*

Theorem 5 rules out the possibility of obtaining better bounds by means of the smoothness framework of [20], or by means of *any* approach aiming at identifying the $\mathbf{b}_i'$ required by Theorem 3, including [8]. These are essentially the only known techniques for obtaining upper bounds on the Bayesian Price of Anarchy. Thus, any improvement on our upper bound for the Discriminatory Auction must use either specific properties of the (Bayes-Nash equilibrium) distribution $\mathcal{D}$, or a completely new approach altogether. The same holds for improvements of the upper bound for the Uniform Price Auction below 2 – and towards the only known lower bound of $\frac{e}{e-1}$ from [14] (should it be worst-case).

Finally, in **Appendix C** we also provide an example, using a discretized strategy space, demonstrating that Bayes-Nash equilibria are not always efficient in the Discriminatory Auction.

4 Smoothness and its Implications

We elaborate on the connections of our results to the smoothness framework for auction mechanisms, which has very recently been developed by Syrgkanis and Tardos [20]. Due to lack of space, all proofs are deferred to **Appendix D**.

We first review the smoothness definitions introduced in [20] (adapted to our multi-unit auction setting). As introduced earlier, let $P_i(\mathbf{b})$ refer to the payment of bidder i under bidding profile $\mathbf{b}$.

Definition 2 ([20]) *A mechanism $\mathcal{M}$ is (λ, μ)-*smooth *for $\lambda > 0$ and $\mu \geq 0$ if for any valuation profile $\mathbf{v}$ and for any bidding profile $\mathbf{b}$ there exists a randomized bidding profile $\mathbf{b}'_i = \mathbf{b}'_i(\mathbf{v}, \mathbf{b}_i)$ for each i such that*

$$\sum_{i \in [n]} \mathbb{E}[u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i})] \geq \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{i \in [n]} P_i(\mathbf{b}).$$

In [20] it is shown that if a mechanism is (λ, μ)-smooth, then several results follow automatically. One such result concerns upper bounds on the Price of Anarchy. Another result is that the smoothness property is retained under *simultaneous and sequential compositions*. In these compositions there are m mechanisms with separate allocation and payment rules. Every bidder specifies for each mechanism a bidding profile. In the simultaneous composition, these profiles are submitted simultaneously, while in the sequential composition, they are submitted sequentially. A bidder expresses his valuation for the m-tuples of outcomes of the mechanisms in a restricted way.[3] We summarize the main composition results of Syrgkanis and Tardos [20] in the theorem below.

Theorem 6 (Theorems 4.2, 4.3, 5.1, and 5.2 in [20])

(i) If $\mathcal{M}$ is (λ, μ)-smooth, then the correlated (or mixed Bayesian) Price of Anarchy of $\mathcal{M}$ is at most $\max\{1, \mu\}/\lambda$.

(ii) If $\mathcal{M}$ is a simultaneous (respectively, sequential) composition of m (λ, μ)-smooth mechanisms, then $\mathcal{M}$ is (λ, μ)-smooth (resp., $(\lambda, \mu + 1)$-smooth).

By exploiting our Key Lemma, we can show that the Discriminatory Auction is smooth. Theorem 7 in combination with Theorem 6 leads to the composition results stated in Table 2 (these bounds are achieved for $\alpha = 1$).

[3] More precisely, in the simultaneous composition it is assumed that the valuation function of each bidder is *fractionally subadditive* across the m mechanisms (see [20] for formal definitions). In the sequential composition, the valuation function of each bidder is defined as the maximum of his valuations over these mechanisms.

Valuation Functions	Discriminatory Auction *Simultaneous*	*Sequential*	Uniform Price Auction *Simultaneous*	*Sequential*
Submodular	$\frac{e}{(e-1)}$	$\frac{2e}{(e-1)}$	3.1462	
Subadditive	$\frac{2e}{(e-1)}$	$\frac{4e}{(e-1)}$	6.2924	

Table 2: Upper bounds on the Bayesian Price of Anarchy for compositions of Discriminatory and Uniform Price Auctions.

Theorem 7 *The Discriminatory Auction is (λ, μ)-smooth (both in the standard and uniform bidding format) with*
(i) $(\lambda, \mu) = (\alpha\left(1 - e^{-1/\alpha}\right), \alpha)$ *for submodular valuation functions, and*
(ii) $(\lambda, \mu) = (\frac{\alpha}{2}\left(1 - e^{-1/\alpha}\right), \alpha)$ *for subadditive valuation functions.*

For auction mechanisms where one needs to impose a no-overbidding assumption, a different smoothness notion is introduced in [20]. Given a mechanism $\mathcal{M}$, define bidder i's *willingness-to-pay* as the maximum payment he could ever pay conditional to being allocated x units, i.e., $B_i(\mathbf{b}_i, x) = \max_{\mathbf{b}_{-i}:x_i(\mathbf{b})=x} P_i(\mathbf{b})$.

Definition 3 ([20]) *A mechanism $\mathcal{M}$ is* weakly *(λ, μ_1, μ_2)-smooth for $\lambda > 0$ and $\mu_1, \mu_2 \geq 0$ if for any valuation profile $\mathbf{v}$ and for any bidding profile $\mathbf{b}$ there exists a randomized bidding profile $\mathbf{b}'_i = \mathbf{b}'_i(\mathbf{v}, \mathbf{b}_i)$ for each bidder i such that*

$$\sum_{i\in[n]} \mathbb{E}[u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i})] \geq \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu_1 \sum_{i\in[n]} P_i(\mathbf{b}) - \mu_2 \sum_{i\in[n]} B_i(\mathbf{b}_i, x_i(\mathbf{b})).$$

Syrgkanis and Tardos [20] establish the following results.

Theorem 8 (Theorems 7.4, C.4 and C.5 in [20])
(i) If $\mathcal{M}$ is (λ, μ_1, μ_2)-weakly smooth, then the correlated (or mixed Bayesian) Price of Anarchy of $\mathcal{M}$ is at most $(\mu_2 + \max\{1, \mu_1\})/\lambda$.
(ii) If $\mathcal{M}$ is a simultaneous (resp., sequential) composition of m (λ, μ_1, μ_2)-weakly smooth mechanisms, then $\mathcal{M}$ is (λ, μ_1, μ_2)-weakly smooth (resp., $(\lambda, \mu_1 + 1, \mu_2)$-weakly smooth).

Using our Key Lemma, we can show that the Uniform Price Auction is weakly smooth. As a consequence, we obtain the composition results stated in Table 2 (these bounds are achieved for $\alpha = -1/(W_{-1}(-1/e^2) + 2) \approx 0.87$).

Theorem 9 *The Uniform Price Auction is weakly (λ, μ_1, μ_2)-smooth (both in the standard and uniform bidding format) with*
(i) $(\lambda, \mu_1, \mu_2) = (\alpha\left(1 - e^{-1/\alpha}\right), 0, \alpha)$ *for submodular valuation functions, and*
(ii) $(\lambda, \mu_1, \mu_2) = (\frac{\alpha}{2}\left(1 - e^{-1/\alpha}\right), 0, \alpha)$ *for subadditive valuation functions.*

Some additional results on mechanisms with *budgets* (see [20]) can be inferred from Theorems 7 and 9. We defer further details to the full version of the paper.

5 Conclusions

We derived inefficiency upper bounds in the incomplete information model for the widely popular Discriminatory and Uniform Price Auctions, when bidders have submodular or subadditive valuation functions. Notably, our bounds for subadditive valuation functions already improve upon the ones that were known for submodular bidders [14, 20]. Moreover, for each of the two formats and valuation function classes we considered both the *standard* bidding interface [12, 15] and a practically motivated *uniform* bidding interface. To derive our results, we elaborated on several techniques from the recent literature on *Simultaneous Auctions* [20, 8, 5, 2]. By the recent developments of [20], our bounds for submodular bidders yield improved inefficiency bounds for *simultaneous* and *sequential* compositions of the considered formats. In absence of an indicative lower bound in the incomplete information model, we showed that our upper bound of $\frac{e}{e-1}$ for the Discriminatory Auction with submodular valuation functions is best possible, w.r.t. the currently known proof techniques. Additionally, for the Uniform Price Auction (with submodular bidders), we showed that, proving an upper bound of less than 2, also requires novel techniques; this poses a particularly challenging problem, given the lower bound of $\frac{e}{e-1}$ from [14].

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APPENDIX A: Pure Nash Equilibria

In this section we discuss the properties of pure Nash equilibria of the two multi-unit auction formats, under both the standard and uniform bidding interfaces. As we show, pure Nash equilibria are always efficient under the Discriminatory Auction, unlike the Uniform Price Auction.

Uniform Pricing

Pure Nash equilibria of the Uniform Price Auction have been analyzed recently in [14]. It is well known that a socially optimum allocation can always be implemented as a pure Nash equilibrium of this auction. Moreoever, by its loose association to the Vickrey Second-Price Auction, the Uniform Price Auction retains certain properties in *undominated strategies*, for bidders with submodular valuation functions. In particular, for any $j \in [k]$, it is always a *weakly dominated strategy* for any bidder i to issue a marginal bid $b_i(j) > m_i(j)$. Moreoever, issuing $b_i(1) \neq v_i(1)$ is also weakly dominated. Markakis and Telelis [14] showed that pure Nash equilibria of the Uniform Price Auction in undominated strategies have inefficiency $\frac{e}{e-1}$. When the Uniform Price Auction is deployed under a *uniform bidding* interface, similar results can be expected, with respect to the inefficiency of (no-overbidding) pure Nash equilibria. In particular:

Lemma 4 *For every pure Nash equilibrium* $\mathbf{b}$ *in undominated strategies, of the Uniform Price Auction with submodular bidders, there exists a pure Nash equilibrium* $\bar{\mathbf{b}}$ *of the auction with uniform bidding, such that* $SW(\bar{\mathbf{b}}) = SW(\mathbf{b})$. *Moreover, for arbitrary valuation functions, every no-overbidding pure Nash equilibrium of the auction with uniform bidding is also a pure Nash equilibrium of the auction under standard bidding, subject to no-overbidding deviations.*

Proof. We may assume that $\mathbf{b}$ has the form prescribed by Proposition 2 in [14]; i.e., for every winning bidder i, $b_i(j) = m_i(j)$, if $j \leq x_i$ and $b_i(j) = 0$ for $j > x_i$. For every losing bidder, $\mathbf{b}_i = (m_i(1), 0, \ldots, 0)$. Then, the uniform price $p(\mathbf{b})$ is equal to $m_i(1) = v_i(1)$ for some i, or 0. We convert $\mathbf{b}$ into a uniform bidding profile $\bar{\mathbf{b}} \equiv \langle \bar{b}_i, q_i \rangle$ as follows: set $\bar{b}_i = v_i(x_i(\mathbf{b}))/x_i(\mathbf{b})$ and $q_i = x_i(\mathbf{b})$ if $x_i(\mathbf{b}) \geq 1$; otherwise set $\bar{b}_i = m_i(1)$ and $q_i = 1$. Then, observe that $p(\bar{\mathbf{b}}) = p(\mathbf{b})$. Because for every *winning* bidder i under $\mathbf{b}$, we have $b_i(x_i(\mathbf{b})) = m_i(x_i(\mathbf{b})) \geq p(\mathbf{b})$, it is also $\bar{b}_i \geq p(\mathbf{b}) = p(\bar{\mathbf{b}})$. Thus, $x_i(\bar{\mathbf{b}}) = x_i(\mathbf{b})$ for all i and $SW(\bar{\mathbf{b}}) = SW(\mathbf{b})$; moreover, $\bar{\mathbf{b}}$ is a pure Nash equilibrium, because $\mathbf{b}$ is one, and the uniform price remains unchanged.

Let $\bar{\mathbf{b}}$ be a no-overbidding pure Nash equilibrium of the Uniform Price Auction under uniform bidding, for arbitrary valuation functions. We argue that it remains a pure Nash equilibrium under the standard bidding. If any losing bidder i has incentive to deviate using a *no-overbidding* standard bid $\mathbf{b}_i$, so as to win at least one additional unit, he may do so also by using a uniform bid $\langle b_i(1), 1 \rangle$, a contradiction. If a winning bidder has an incentive to deviate under $\bar{\mathbf{b}}$ using a *no-overbidding* standard bid $\mathbf{b}_i$, having q_i non-zero components (marginal bids), then he may as well do so with a uniform bid $\langle \sum_{j \leq q_i} b_i(j)/q_i, q_i \rangle$; indeed, because $\mathbf{b}_i$ is submodular, $\sum_{j \leq q_i} b_i(j)/q_i \geq b_i(q_i)$. Thus, the assumed uniform bid should grant i at least as many units as $\mathbf{b}_i$. $\square$

The above Lemma, along with Theorem 2 of [14] and Theorem 1 from Section 3, leads to the following:

Corollary 1 *The Price of Anarchy for pure Nash equilibria of the Uniform Price Auction with submodular bidders under the uniform bidding interface is at least* $\left(1 - \frac{1}{e} + \frac{2}{k}\right)^{-1}$*, for* $k \geq 9$*, and at most* 3.1462*.*

The lower bound follows directly from the first statement of Lemma 4 and the lower bound on the inefficiency of pure Nash equilibria in undominated strategies of [14] (Theorem 2). The upper bound follows from the second statement of Lemma 4 and statement *(ii)* of Theorem 1 (Section 3), which applies for mixed Bayes-Nash equilibria with no-overbidding, thus, also for pure Nash.

For wider valuation function classes than submodular, we do not know whether the Uniform Price Auction generally has pure Nash equilibria. To the best of our knowledge, and as mentioned by Milgrom in [15], the standard multi-unit auction formats have not been studied before, for any larger class of valuation functions. In Section 3 we give upper bounds of 4 and 6.2924 on the Price of Anarchy of *mixed Bayes-Nash* equilibria for *subadditive* valuation functions, under the *standard and uniform bidding interfaces*, respectively. By Lemma 4 however, the former bound of 4 is valid also for uniform bidding, in the case of *pure Nash* equilibria.

Corollary 2 *The Price of Anarchy of pure Nash equilibria of the Uniform Price Auction with subadditive bidders under the standard or the uniform bidding interface is at least* 2 *and at most* 4*.*

The upper bound of this corollary follows by the second statement of Lemma 4 and by Theorem 2, discussed in Section 3. The lower bound is shown explicitly in Section 3.3.

Discriminatory Pricing

The Discriminatory Auction, as a generalization of the First-Price Auction, is not guaranteed to possess pure Nash equilibria; their existence depends heavily on the choice of a tie-breaking rule, as is often the case for games where players have a continuum of strategies. For example, consider the first-price auction where the valuation of bidder 1 is 1, the valuation of bidder 2 is $\epsilon < 1$, and the tie-breaking rule always favors bidder 2. Obviously there can be no equilibrium where bidder 2 bids above 1. Furthermore, if bidder 2 bids some value $\delta < 1$, then bidder 1 does not have a best response in $(\delta, 1)$; no matter what he bids to win the unit, he always has an incentive to lower his bid while still being above δ. Hence there is no Nash equilibrium for this auction. We exhibit here that, as with first-price auctions, an appropriate choice of a tie-breaking rule induces a *uniform bidding* profile that is a pure Nash equilibrium for the auction, even subject to deviations under the standard bidding interface. Additionally, we show that we can always obtain close approximations to pure Nash equilibria, i.e., pure ϵ-equilibria, for every possible tie breaking rule.

Proposition 1

(i) For every Discriminatory Auction there is a tie-breaking rule inducing a uniform bidding profile that is a pure Nash equilibrium under that tie-breaking rule.
(ii) For every $\epsilon > 0$*, the Discriminatory Auction has a pure* ϵ*-equilibrium.*

Proof. (i) Let $\hat{\mathbf{m}}$ be the nk-dimensional vector obtained by appending all vectors $\mathbf{m}_i, i \in [n]$, and let $\tilde{\mathbf{m}}$ be the vector obtained by non-increasingly ordering $\hat{\mathbf{m}}$. We show that the set of bid vectors b where every bidder sets all his marginal bids to $\tilde{m}(k)$ is a pure equilibrium, if ties are broken according to any tie-breaking rule that satisfies $m_i(x_i(\mathbf{b})) \geq \tilde{m}(k)$.

Assume without loss of generality that there are at least 2 players. Let $\mathbf{b}$ be the bidding profile where all players bid $\tilde{m}(k)$ on all items, and break ties in any way that satisfies that $m_i(x_i(\mathbf{b})) \geq \tilde{m}(k)$. To see why this is a pure equilibrium, consider the player deviating to bid vector $\mathbf{b}'_i$. Note that $\mathbf{b}'_i - \mathbf{b}_i$ is non-increasing. Define ℓ as the lowest index such that $(\mathbf{b}_i - \mathbf{b}'_i)(\ell)$ is negative (and define ℓ as $k+1$ if there is no such index). In case $\ell \leq x_i(\mathbf{b})$, then the utility of player i will certainly not increase by deviating to $\mathbf{b}'_i$, as he will lose utility from the fact that $x_i(\mathbf{b}) - \ell$ less items are now allocated to him under $(\mathbf{b}'_i, \mathbf{b}_{-i})$, compared to $\mathbf{b}$. As player i used to derive non-negative utility from these items under $\mathbf{b}$, this removal of items accounts for a non-negative decrease in utility. Moreover, player i increases his bid on his first ℓ items, so this accounts for a non-negative decrease in utility as well. His total utility will therefore decrease in this case.

In case $\ell > x_i(\mathbf{b})$, we are in a situation where player i increases his bids (under $\mathbf{b}'_i$, compared to $\mathbf{b}_i$) on some of the first $x_i(\mathbf{b})$ items by at least 0, so he will win these items under $(\mathbf{b}'_i, \mathbf{b}_{-i})$ but spend more money on it, leading to a decrease in utility. On any remaining items that player i wins under $(\mathbf{b}'_i, \mathbf{b}_{-i})$, he overbids. This also accounts for a non-negative decrease in utility. The total decrease in utility is thus non-negative in this case.

(ii) Let $\mathbf{b}^*$ be a social welfare maximizing bidding profile. Consider the uniform bidding profile $\bar{\mathbf{b}}$ as defined in Proposition 1 (i). Let $(\xi_i)_{i\in N}$ be a set of vectors that indicate the optimal allocation $(x_i(\mathbf{b}^*))_{i\in N}$, i.e., ξ_i is the $(0,1)$-vector of which the first $x_i(\mathbf{b}^*)$ entries are 1, and the remaining entries are 0. We show that $\tilde{\mathbf{b}} = \mathbf{b} + \epsilon\xi/k$ is a pure ϵ-equilibrium. The reasoning we apply is largely analogous to the proof of Proposition 1 (i).

First of all, observe that there are no ties that need to be broken under $\tilde{\mathbf{b}}$, and that the allocation $(x_i(\tilde{\mathbf{b}})_{i\in N}$ satisfies $m_i(x_i(\mathbf{b})) \geq \tilde{m}(k)$.

Consider the player deviating to bid vector $\mathbf{b}'_i$. If player i wins less items under $(\mathbf{b}'_i, \tilde{\mathbf{b}}_{-i})$ than under $\mathbf{b}$, he will experience an increase in utility of at most $(x_i(\tilde{\mathbf{b}}) - x_i(\mathbf{b}'))\epsilon/k$ due to losing items, because the utility that player i derived under $\mathbf{b}'$ from each of these lost items was at least $-\epsilon/k$. On the remaining items that player i still wins, the player increases his bid by at least $-\epsilon/k$, and this accounts for an increase in utility of at most $x_i(\mathbf{b}')\epsilon/k$. The total increase in utility is thus at most $x_i(\mathbf{b}')\epsilon/k \leq \epsilon$.

If player i wins at least as much items under $(\mathbf{b}'_i, \tilde{\mathbf{b}}_{-i})$ than under $\tilde{\mathbf{b}}$, the player will have increased his bids on the first $x_i(\tilde{\mathbf{b}})$ items by at least $-\epsilon/k$, and by at least 0 on the remaining items. For these remaining items, the player experiences non-positive utility under $(\mathbf{b}'_i, \mathbf{b}_{-i})$, whereas he experienced 0 utility under $\tilde{\mathbf{b}}$. Therefore, the total increase in utility is in this case at most $x_i(\tilde{\mathbf{b}})\epsilon/k \leq \epsilon$. □

We show next that, whenever pure Nash equilibria exist, they are socially optimal, even with arbitrary valuation functions. This is in analogy with other results on mechanisms with first price rules [11].

Theorem 10 *Pure Nash equilibria of the Discriminatory Auction (with the standard or the uniform bidding interface) are always efficient, even for bidders with arbitrary valuation functions.*

The proof of Theorem 10 is based on the following Lemma, which captures the main properties of pure Nash equilibria. Notice that, the first in the Lemma below, essentially states that every pure Nash equilibrium of the auction occurs at a *uniform bidding* profile. Thus, the theorem is also valid for the uniform bidding interface.

Lemma 5 *Let* $\mathbf{b}$ *be a pure Nash equilibrium in a given Discriminatory Auction where the bidders have general valuation functions. Let* $d = \max\{b_i(j) : i \in [n], j \in [k], j > x_i(\mathbf{b})\}$*. Then:*

(i) For any bidder i *who wins at least one item under* $\mathbf{b}$*, and for all* $j \in [x_i(\mathbf{b})]$*,* $b_i(j) = d$*,*

(ii) $\ell d \leq \sum_{j=x_i(\mathbf{b})-\ell+1}^{x_i(\mathbf{b})} m_i(j)$ *for all* $i \in [n]$*, and* $\ell \in [x_i(\mathbf{b})]$*,*

(iii) $\sum_{j=x_i(\mathbf{b})+1}^{x_i(\mathbf{b})+\ell} m_i(j) \leq \ell d$ *for all* $i \in [n]$*, and* $\ell \in [k - x_i(\mathbf{b})]$*.*

Proof. Let c be the smallest value in $\{b_i(j) : i \in [n], j \in [k], j \leq x_i(\mathbf{b})\}$, i.e., the smallest winning marginal bid. Observe that $c = d$: Otherwise, a player i that bids $b_i(x_i(\mathbf{b})) = c$ could change $b_i(x_i(\mathbf{b}))$ to a lower bid in order to obtain more utility. For the same reasons, we conclude that any winning marginal bid $b_i(j)$ is equal to the largest marginal bid that is smaller than $b_i(j)$. It follows inductively that all winning marginal bids are equal to d. This establishes point (1) of the claim.

Suppose that for some $i \in [n]$ and $\ell \in [x_i(\mathbf{b})]$, it holds that $\ell d = \sum_{j=x_i(\mathbf{b})-\ell+1}^{x_i(\mathbf{b})} b_i(j) > \sum_{j=x_i(\mathbf{b})-\ell+1}^{x_i(\mathbf{b})} m_i(j)$. Then if player i changes all marginal bids $b_i(j)$ for $j \in \{j : \ell \leq j\}$ to 0, he would increase his utility. This is not possible since $\mathbf{b}$ is a pure equilibrium, so we conclude that for all $i \in [n]$ and $\ell \in [x_i(\mathbf{b})]$, it holds that $\sum_{j=x_i(\mathbf{b})-\ell+1}^{x_i(\mathbf{b})} b_i(j) \leq \sum_{j=x_i(\mathbf{b})-\ell+1}^{x_i(\mathbf{b})} m_i(j)$. This establishes point (2) of the claim.

Note that there is no $i \in [n]$ and $\ell \in [k - x_i(\mathbf{b})]$ such that $\sum_{j=x_i(\mathbf{b})+1}^{x_i(\mathbf{b})+\ell} m_i(j) > \ell d$: Otherwise, if player i would change his marginal bids $b_i(j), j \in \{j : 1 \leq j \leq x_i(\mathbf{b}) + \ell\}$ to $d + \epsilon$ for some $\epsilon > 0$, then player i's utility increases by $\sum_{j=x_i(\mathbf{b})+1}^{x_i(\mathbf{b})+\ell} m_i(j) - \ell(d + \epsilon) - x_i(\mathbf{b})\epsilon$. Because $\sum_{j=x_i(\mathbf{b})+1}^{x_i(\mathbf{b})+\ell} m_i(j) - \ell d$ is positive, this total increase is positive when we take for ϵ a sufficiently small value. This is in contradiction with the fact that $\mathbf{b}$ is a pure equilibrium, and this establishes point (3) of the claim. □

Proof. [of Theorem 10] Let $\mathbf{b}^*$ be a bid vector that attains the optimum social welfare. Denote by A the set of bidders that get more items under $\mathbf{b}$ than under $\mathbf{b}^*$. For a bidder $i \in A$, define ℓ_i as the number of extra items that i gets under $\mathbf{b}$, when compared to $\mathbf{b}^*$; i.e., $\ell_i = x_i(\mathbf{b}) - x_i(\mathbf{b}^*)$. Denote by B the set of bidders that get more items under $\mathbf{b}^*$ than under $\mathbf{b}$. For a bidder $i \in B$, define ℓ_i as the number of extra items that i gets under $\mathbf{b}^*$, when compared to $\mathbf{b}$; i.e., $\ell_i = x_i(\mathbf{b}^*) - x_i(\mathbf{b})$. Then,

$$\sum_{i=1}^{n} v_i(x_i(\mathbf{b})) - \sum_{i=1}^{n} v_i(x_i(\mathbf{b}^*)) = \sum_{i=1}^{n} \left(\sum_{j=1}^{x_i(\mathbf{b})} m_i(j) - \sum_{j=1}^{x_i(\mathbf{b}^*)} m_i(j) \right)$$

$$= \sum_{i \in A} \sum_{j=x_i(\mathbf{b})-\ell_i+1}^{x_i(\mathbf{b})} m_i(j) - \sum_{i \in B} \sum_{j=x_i(\mathbf{b})+1}^{x_i(\mathbf{b})+\ell_i} m_i(j) \geq \sum_{i \in A} \ell_i d - \sum_{i \in B} \ell_i d = 0.$$

The inequality in the derivation above follows from points *(ii)* and *(iii)* of Lemma 5, and the final equality holds because $\sum_{i \in A} \ell_i = \sum_{i \in B} \ell_i$. Thus, the social welfare of the pure equilibrium $\mathbf{b}$ is optimal. □

APPENDIX B: Ommitted Proofs from Section 3

Proof of Theorem 3

For any particular bidder $i \in [n]$ let $\mathcal{U}^i \subseteq \mathcal{V}$ denote the subset of valuation profiles $\mathbf{v} \in \mathcal{V}$ where $x^{\mathbf{v}}(i) \geq 1$, i.e., $\mathcal{U}^i = \{\mathbf{v} \in \mathcal{V} | x^{\mathbf{v}}(i) \geq 1\}$; these are the profiles under which i is a "socially optimum winner". Accordingly, we let $\mathcal{W}^{\mathbf{v}}$ denote the subset of "socially optimum winners" in valuation profile $\mathbf{v} \in \mathcal{V}$.

Consider a Bayes-Nash equilibrium $\mathbf{B}$. Fix any valuation profile $\mathbf{v} = (\mathbf{v}_i, \mathbf{v}_{-i}) \in \mathcal{V}$ and a bidder $i \in [n]$. Assume that bidder i deviates according to a bidding vector $\mathbf{b}'_i$ satisfying (1), taking $\mathcal{P}_{-i} = \mathbf{B}_{-i}^{\mathbf{w}_{-i}}, \mathbf{w}_{-i} \sim \pi_{-i}$. Taking expectation over all valuation profiles $\mathbf{w}_{-i} \in \mathcal{V}_{-i}$, we obtain

$$\begin{aligned}
\mathop{\mathbb{E}}_{\mathbf{w}_{-i}|\mathbf{v}_i}\left[\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}^{\mathbf{w}_{-i}}}\left[u_i^{\mathbf{v}_i}(\mathbf{b}'_i,\mathbf{b}_{-i})\right]\right] &\geq \lambda v_i(x_i^{\mathbf{v}}) - \mu \cdot \mathop{\mathbb{E}}_{\mathbf{w}_{-i}|\mathbf{v}_i}\left[\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}^{\mathbf{w}_{-i}}}\left[\sum_{j=1}^{x_i^{\mathbf{v}}}\beta_j(\mathbf{b}_{-i})\right]\right] \\
&\geq \lambda v_i(x_i^{\mathbf{v}}) - \mu \cdot \mathop{\mathbb{E}}_{\mathbf{w}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{w}}}\left[\sum_{j=1}^{x_i^{\mathbf{v}}}\beta_j(\mathbf{b})\right]\right],
\end{aligned}$$

where the last inequality holds because $\beta_j(\mathbf{b}_{-i}) \leq \beta_j(\mathbf{b})$ for every $j = 1, \ldots, k$ and by the independence of π_i, i.e., $\sum_{\mathbf{w}_{-i}} \pi(\mathbf{w}_{-i}|v_i) = 1 = \sum_{\mathbf{w}} \pi(\mathbf{w})$. Because $\mathbf{B}$ is a Bayes-Nash equilibrium, bidder i does not have an incentive to deviate and thus

$$\mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i}\left[\mathbb{E}_{\mathbf{b}\sim\mathbf{B}^{(\mathbf{v}_i,\mathbf{w}_{-i})}}\left[u_i^{\mathbf{v}_i}(\mathbf{b})\right]\right] \geq \mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i}\left[\mathbb{E}_{\mathbf{b}_{-i}\sim\mathbf{B}_{-i}^{\mathbf{w}_{-\mathbf{i}}}}\left[u_i^{\mathbf{v}_i}(\mathbf{b}'_i,\mathbf{b}_{-i})\right]\right].$$

We conclude that

$$\mathbb{E}_{\mathbf{w}_{-i}|\mathbf{v}_i}\left[\mathbb{E}_{\mathbf{b}\sim\mathbf{B}^{(\mathbf{v}_i,\mathbf{w}_{-i})}}\left[u_i^{\mathbf{v}_i}(\mathbf{b})\right]\right] + \mu\mathbb{E}_{\mathbf{w}}\left[\mathbb{E}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{w}}}\left[\sum_{j=1}^{x_i^{\mathbf{v}}}\beta_j(\mathbf{b})\right]\right] \geq \lambda v_i(x_i^{\mathbf{v}}).$$

Taking expectation of both sides over the distribution of $\mathbf{v} \in \mathcal{U}^i$ and summing over all bidders we obtain

$$\begin{aligned}
&\sum_{i\in[n]}\sum_{\mathbf{v}\in\mathcal{U}^i}\pi(\mathbf{v})\cdot\left(\mathop{\mathbb{E}}_{\mathbf{w}_{-i}|\mathbf{v}_i}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{(v_i,\mathbf{w}_{-i})}}\left[u_i^{v_i}(\mathbf{b})\right]\right] + \mu\mathop{\mathbb{E}}_{\mathbf{w}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{w}}}\left[\sum_{j=1}^{x_i^{\mathbf{v}}}\beta_j(\mathbf{b})\right]\right]\right) \\
&\geq \sum_{i\in[n]}\sum_{\mathbf{v}\in\mathcal{U}^i}\pi(\mathbf{v})\cdot\lambda v_i(x_i^{\mathbf{v}}) = \sum_{\mathbf{v}}\pi(\mathbf{v})\sum_{i\in\mathcal{W}^{\mathbf{v}}}\lambda v_i(x_i^{\mathbf{v}}) = \lambda\mathbb{E}_{\mathbf{v}}[SW(\mathbf{v},\mathbf{x}^{\mathbf{v}})].
\end{aligned}$$

By standard manipulations, the latter simplifies to

$$\mathop{\mathbb{E}}_{\mathbf{v}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{v}}}\left[\sum_{i\in[n]}u_i^{\mathbf{v}_i}(\mathbf{b})\right]\right] + \mu\mathop{\mathbb{E}}_{\mathbf{w}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{w}}}\left[\sum_{i\in[n]}\sum_{j=1}^{x_i^{\mathbf{v}}}\beta_j(\mathbf{b})\right]\right] \geq \lambda\mathop{\mathbb{E}}_{\mathbf{v}}[SW(\mathbf{v},\mathbf{x}^{\mathbf{v}})].$$

Note that $\sum_{i\in[n]} x_i^{\mathbf{v}} = k$ and that $\beta_j(\mathbf{b})$ is non-decreasing in j. We can therefore bound

$$\sum_{i\in[n]}\sum_{j=1}^{x_i^{\mathbf{v}}}\beta_j(\mathbf{b}) \leq \sum_{j=1}^{k}\beta_j(\mathbf{b})$$

and obtain

$$\mathop{\mathbb{E}}_{\mathbf{v}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{v}}}\left[\sum_{i\in[n]} u_i^{\mathbf{v}_i}(\mathbf{b})\right]\right] + \mu\mathop{\mathbb{E}}_{\mathbf{w}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{w}}}\left[\sum_{j=1}^{k}\beta_j(\mathbf{b})\right]\right] \geq \lambda\mathop{\mathbb{E}}_{\mathbf{v}}[SW(\mathbf{v},\mathbf{x}^{\mathbf{v}})]. \tag{3}$$

Note that for the discriminatory pricing rule the total payments under $\mathbf{b}$ are equal to $\sum_{j=1}^{k}\beta_j(\mathbf{b})$. Thus (3) yields

$$\mathop{\mathbb{E}}_{\mathbf{v}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{v}}}[SW(\mathbf{v},\mathbf{b})]\right] + (\mu-1)\mathop{\mathbb{E}}_{\mathbf{w}}\left[\mathop{\mathbb{E}}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{w}}}\left[\sum_{j=1}^{k}\beta_j(\mathbf{b})\right]\right] \geq \lambda\mathop{\mathbb{E}}_{\mathbf{v}}[SW(\mathbf{v},\mathbf{x}^{\mathbf{v}})].$$

If $\mu \leq 1$, the first statement of the theorem holds. If $\mu > 1$, then we exploit that the total payments satisfy $\sum_{j=1}^{k}\beta_j(\mathbf{b}) \leq \sum_{i\in[n]} v_i(x_i(\mathbf{b})) = SW(\mathbf{v},\mathbf{b})$ because players never overbid.[4] Dividing both sides by $\mu > 0$ proves the first statement of the theorem in this case.

For the uniform pricing rule we use that $\sum_{j=1}^{k}\beta_j(\mathbf{b}) \leq \sum_{i\in[n]} v_i(x_i(\mathbf{b})) = SW(\mathbf{v},\mathbf{b})$ because of the no-overbidding assumption and that $\sum_{i\in[n]} u_i^{\mathbf{v}_i}(\mathbf{b}) \leq \sum_{i\in[n]} v_i(x_i(\mathbf{b})) = SW(\mathbf{v},\mathbf{b})$. Thus (3) yields

$$(\mu+1)\mathbb{E}_{\mathbf{v}}\left[\mathbb{E}_{\mathbf{b}\sim\mathbf{B}^{\mathbf{v}}}[SW(\mathbf{v},\mathbf{b})]\right] \geq \lambda\mathbb{E}_{\mathbf{v}}[SW(\mathbf{v},\mathbf{x}^{\mathbf{v}})].$$

Dividing both sides by $\mu + 1 > 0$ proves the second statement of the theorem. □

Proof of Lemma 2

Recall that $\tau_i = \arg\min_{j\in[x_i^{\mathbf{v}}]} v_i(j)/j$. Thus, it suffices to show that $v_i(\tau_i)/\tau_i \geq v_i(x_i^{\mathbf{v}})/(2x_i^{\mathbf{v}})$. By subadditivity, $v_i(x_i^{\mathbf{v}})/(2x_i^{\mathbf{v}})$ is at most:

$$\frac{1}{2}\left(\frac{v_i(x_i^{\mathbf{v}}-\tau_i)}{x_i^{\mathbf{v}}} + \frac{v_i(\tau_i)}{x_i^{\mathbf{v}}}\right) \leq \begin{cases} \frac{1}{2}\left(\frac{v_i(x_i^{\mathbf{v}}-\tau_i)}{\tau_i} + \frac{v_i(\tau_i)}{\tau_i}\right) \leq \frac{v_i(\tau_i)}{\tau_i} & \text{if } x_i^{\mathbf{v}} \leq 2\tau_i \\ \frac{1}{2}\left(\frac{v_i(x_i^{\mathbf{v}}-\tau_i)}{x_i^{\mathbf{v}}} + \frac{v_i(\tau_i)}{\tau_i}\right) \leq \frac{v_i(\tau_i)}{\tau_i} & \text{if } x_i^{\mathbf{v}} > 2\tau_i \end{cases}$$

where in the first case we used monotonicity, i.e., $v_i(x_i^{\mathbf{v}} - \tau_i) \leq v_i(\tau_i)$ and in the second case we used subadditivity, i.e., $\frac{v_i(\tau_i)}{\tau_i} \geq \frac{v_i(x_i^{\mathbf{v}}-\tau_i)}{x_i^{\mathbf{v}}-\tau_i+\tau_i} = \frac{v_i(x_i^{\mathbf{v}}-\tau_i)}{x_i^{\mathbf{v}}}$. □

Proof of Lemma 3

Discriminatory Pricing

We consider first the discriminatory pricing rule. We shall re-use certain arguments from the proof for this case, in the proof for uniform pricing. Fix any bidder i and let $\mathbf{b}_i$ be a bidding vector, conforming to the requirement of non-increasing marginal bids and having only its first x components equal to a non-zero value. Given any bidding profile $\mathbf{b}_{-i}$:

$$u_i^{v_i}(\mathbf{b}_i,\mathbf{b}_{-i}) \geq v_i\Big(x_i(\mathbf{b}_i,\mathbf{b}_{-i})\Big) - \sum_{j\leq x} b_i(j),$$

[4] In order to derive the claimed bound for the case $\mu > 1$ we need to exploit the no-overbidding assumption mentioned in Section 2 for the Discriminatory Auction as well. However, all our bounds derived in this paper exploit only that $\mu \leq 1$ and thus hold even without this assumption.

because i may pay at most $\sum_{j\leq k} b_i(j) = \sum_{j\leq x} b_i(j)$, by the definition of $\mathbf{b}_i$ and the auction's payment rule. Taking expectation over the distribution $\mathcal{P}$ of $\mathbf{b}_{-i}$ we have:

$$\mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}}\Big[u_i^{v_i}(\mathbf{b}_i,\mathbf{b}_{-i})\Big] \geq \mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}}\Big[v_i\Big(x_i(\mathbf{b}_i,\mathbf{b}_{-i})\Big)\Big] - \sum_{j\leq x} b_i(j) \tag{4}$$

From this point on we analyze the right-hand side of (4). Given the distribution $\mathcal{P}$ of $\mathbf{b}_{-i}$, let $\mathcal{D}$ denote the distribution of the k highest bids in $\mathbf{b}_{-i}$, $\beta_1(\mathbf{b}_{-i}) \leq \cdots \leq \beta_k(\mathbf{b}_{-i})$; to simplify notation, in the sequel we use simply β_j, $j = 1,\ldots,k$, without a reference to $\mathbf{b}_{-i}$. For every fixed bid vector of bidder i, the expected utility of i when the other bidders bid according to $\mathcal{P}$, is equal to the expected utility of bidder i in the two-bidders auction, where the other bidder bids according to $\mathcal{D}$. We can thus assume that i competes only against $\beta\sim\mathcal{D}$, containing β_j, $j = 1,\ldots,k$. For notational purposes below, we shall also use γ to denote a vector drawn from $\mathcal{D}$.

We consider what happens when i responds to $\mathcal{P}$ (i.e., $\mathcal{D}$, in the two-bidders auction), by bidding a $\mathbf{b}_i = \tilde{\beta}$, that he constructs as follows: he samples a vector β from $\mathcal{D}$ and *zeroes-out* the $k-x$ *highest* values in β. Subsequently, he adds to all components of the "truncated" vector a sufficiently small ϵ. This latter modification we shall omit in our analysis below, to simplify its exposition. It serves the purpose of avoiding ties with the opposing bids; the analysis is valid when this ϵ becomes vanishingly small. Let $\tilde{\mathcal{D}}$ denote the distribution of such $\mathbf{b}_i$. Continuing from (4), the expected utility of i over $\mathbf{b}_i\sim\tilde{\mathcal{D}}$ is:

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\mathbf{b}_i\sim\tilde{\mathcal{D}}}\left[\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathcal{P}}\Big[u_i^{v_i}(\mathbf{b}_i,\mathbf{b}_{-i})\Big]\right] \geq \mathop{\mathbb{E}}_{\mathbf{b}_i\sim\tilde{\mathcal{D}}}\left[\mathop{\mathbb{E}}_{\mathbf{b}_{-i}\sim\mathcal{P}}\Big[v_i\Big(x_i(\mathbf{b}_i,\mathbf{b}_{-i})\Big)\Big] - \sum_{j\leq x} b_i(j)\right] \\
&= \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \mathbf{b}_{-i}\sim\mathcal{P}}}\left[v_i\Big(x_i(\tilde{\beta},\mathbf{b}_{-i})\Big) - \sum_{j\leq x}\beta_j\right] = \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\gamma)\Big)\right] - \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[\sum_{j\leq x}\beta_j\right] \qquad (5) \\
&= \frac{1}{2}\cdot 2\cdot \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\gamma)\Big)\right] - \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[\sum_{j\leq x}\beta_j\right] \\
&= \frac{1}{2}\left\{\mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\gamma)\Big) + v_i\Big(x_i(\tilde{\gamma},\beta)\Big)\right]\right\} - \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[\sum_{j\leq x}\beta_j\right] \\
&\geq \frac{1}{2}\mathbb{E}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\Big[v_i(x)\Big] - \mathbb{E}_{\beta\sim\mathcal{D}}\left[\sum_{j\leq x}\beta_j\right] = \frac{1}{2}v_i(x) - \mathbb{E}_{\mathbf{b}_{-i}\sim\mathcal{P}}\left[\sum_{j=1}^{x}\beta_j(\mathbf{b}_{-i})\right] \qquad (6)
\end{aligned}$$

where (5) is due to the fact that $\sum_{j\leq x} b_i(j) = \sum_{j\leq x}\tilde{\beta}_j = \sum_{j\leq x}\beta_j$, by construction of $\tilde{\beta}$ and because $\mathbf{b}_i$ has its components (taken from $\tilde{\beta}$) in non-increasing order. The last inequality, in (6), holds by subadditivity of v_i, particularly, because $x_i(\beta,\gamma) + x_i(\gamma,\beta) \geq x$ (when ties are always resolved in favor of i). $\square$

Uniform Pricing

Fix any bidder i and let $\mathbf{b}_i$ be a bidding vector with non-zero value for each of the first x components and zero value for the rest. Notice that, given any bidding profile $\mathbf{b}_{-i}$:

$$\begin{aligned} u_i^{v_i}(\mathbf{b}_i, \mathbf{b}_{-i}) &= v_i\Big(x_i(\mathbf{b}_i, \mathbf{b}_{-i})\Big) - x_i(\mathbf{b}_i, \mathbf{b}_{-i}) \cdot p(\mathbf{b}_i, \mathbf{b}_{-i}) \\ &\geq v_i\Big(x_i(\mathbf{b}_i, \mathbf{b}_{-i})\Big) - \sum_{j \leq x} b_i(j) \end{aligned}$$

where $p(\mathbf{b}_i, \mathbf{b}_{-i})$ is the uniform price that i will pay under the profile $(\mathbf{b}_i, \mathbf{b}_{-i})$. This price cannot be more than the sum of the winning bids, $\sum_{j \leq x} b_i(j)$; this justifies the inequality above. Taking expectation over the distribution $\mathcal{P}$ of $\mathbf{b}_{-i}$ we have:

$$\mathbb{E}_{\mathbf{b}_{-i} \sim \mathcal{P}}\Big[u_i^{v_i}(\mathbf{b}_i, \mathbf{b}_{-i})\Big] \geq \mathbb{E}_{\mathbf{b}_{-i} \sim \mathcal{P}}\Big[v_i\Big(x_i(\mathbf{b}_i, \mathbf{b}_{-i})\Big)\Big] - \sum_{j \leq x} b_i(j) \tag{7}$$

Our analysis from this point on will focus on identifying an appropriate bid $\mathbf{b}_i$ for bidder i, that satisfies also *no-overbidding*. Subsequently, we shall return to process (7) further. As previously, given the distribution $\mathcal{P}$ of $\mathbf{b}_{-i}$, we define the distribution $\mathcal{D}$ of the k highest bids in $\mathbf{b}_{-i}$, $\beta_1(\mathbf{b}_{-i}) \leq \cdots \leq \beta_k(\mathbf{b}_{-i})$; we use simply β_j, $j = 1, \ldots, k$, without a reference to $\mathbf{b}_{-i}$. Each of these is a potential uniform price that i might need to pay, depending on his own bid $\mathbf{b}_i$. Moreover, we can assume that i competes against a bid vector β, containing only the bids β_j, $j = 1, \ldots, k$, and need not bother with any smaller bids. Fix any such vector β from the support of $\mathcal{D}$ and let $T_\beta \subseteq [x]$ be a maximal subset of indices, such that: $\sum_{j \in T_\beta} \beta_j > v_i(|T_\beta|)$. Let $\bar{T}_\beta = [x] \setminus T_\beta$. Then, *we claim that:* $\sum_{j \in \bar{T}_\beta} \beta_j \leq v_i(|\bar{T}_\beta|)$. Indeed, if there exists $R \subseteq \bar{T}_\beta$ with $\sum_{j \in R} \beta_j > v_i(|R|)$, then, by subadditivity of v_i and monotonicity:

$$v_i(|R \cup T_\beta|) \leq v_i(|R|) + v_i(|T_\beta|) < \sum_{j \in R} \beta_j + \sum_{j \in T_\beta} \beta_j = \sum_{j \in R \cup T_\beta} \beta_j$$

which contradicts the maximality of T_β. Next, define $\tilde{\mathcal{D}}$ to be a distribution of bidding vectors similar to $\mathcal{D}$, differing from it as follows. For every vector β in the support of $\mathcal{D}$ that occurs with a certain probability, a vector $\tilde{\beta}$ exists in the support of $\tilde{\mathcal{D}}$, that occurs with the same probability and is made from β according to the following rules:

1. Identify a subset of indices T_β for β and let $\bar{T}_\beta = [x] \setminus T_\beta$.

2. $\tilde{\beta}$ contains all bids of β except for $\{\beta_j | j \in T_\beta\}$; it contains a zero for each bid therein.

As in the previous proof for the Discriminatory Auction, a further modification that we shall omit in our analysis is that: i adds a sufficiently small ϵ to each of the components of the resulting vector. This serves the purpose of breaking ties in favor of i, as before. Sampling a vector from $\tilde{\mathcal{D}}$ is equivalent to sampling a vector β from $\mathcal{D}$ and constructing $\tilde{\beta}$ as prescribed (using "$\tilde{\ }$" over a β will correspond to this processing). For any $\beta \sim \mathcal{D}$ and for any arbitrary bid $\mathbf{b}_i$, we observe that $x_i(\mathbf{b}_i, \tilde{\beta}) \leq x_i(\mathbf{b}_i, \beta) + |T_\beta|$. Thus, $v_i(x_i(\mathbf{b}_i, \tilde{\beta})) \leq v_i(x_i(\mathbf{b}_i, \beta)) + v_i(|T_\beta|)$, by subadditivity and monotonicity of v_i. Using $\sum_{j \leq x} \beta_j - \sum_{j \in \bar{T}_\beta} \beta_j = \sum_{j \in T_\beta} \beta_j \geq v_i(|T_\beta|)$, we obtain:

$$v_i\Big(x_i(\mathbf{b}_i, \beta)\Big) - \sum_{j \in \bar{T}_\beta} \beta_j \geq v_i\Big(x_i(\mathbf{b}_i, \tilde{\beta})\Big) - \sum_{1 \leq j \leq x} \beta_j \tag{8}$$

$$\text{Thus:} \quad \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[v_i\Big(x_i(\mathbf{b}_i,\beta)\Big) - \sum_{j\in\bar{T}_\beta}\beta_j\right] \geq \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[v_i\Big(x_i(\mathbf{b}_i,\tilde{\beta})\Big) - \sum_{1\leq j\leq x}\beta_j\right] \tag{9}$$

Now consider $\mathbf{b}_i$ being drawn from the distribution $\tilde{\mathcal{D}}$. Notice that, by their construction, all the bid vectors in the support of $\tilde{\mathcal{D}}$ satisfy no-overbidding. We take expectation of the *left-hand side* of (9) over $\mathbf{b}_i \sim \tilde{\mathcal{D}}$:

$$\begin{aligned}\mathop{\mathbb{E}}_{\substack{\mathbf{b}_i\sim\tilde{\mathcal{D}}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\mathbf{b}_i,\gamma)\Big) - \sum_{j\in\bar{T}_\gamma}\gamma_j\right] &= \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\gamma)\Big) - \sum_{j\in\bar{T}_\gamma}\gamma_j\right]\\ &= \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\gamma)\Big)\right] - \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[\sum_{j\in\bar{T}_\beta}\beta_j\right]\end{aligned} \tag{10}$$

Accordingly, we take expectation of the *right-hand side* of (9) over $\bar{\mathcal{D}}$, to obtain:

$$\mathop{\mathbb{E}}_{\substack{\mathbf{b}_i\sim\bar{\mathcal{D}}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\mathbf{b}_i,\tilde{\gamma})\Big) - \sum_{1\leq j\leq x}\gamma_j\right] = \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\tilde{\gamma})\Big)\right] - \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[\sum_{1\leq j\leq x}\beta_j\right] \tag{11}$$

Finally, we take expectation of (7) over $\mathbf{b}_i \sim \tilde{\mathcal{D}}$, to obtain:

$$\begin{aligned}&\mathop{\mathbb{E}}_{\substack{\mathbf{b}_i\sim\tilde{\mathcal{D}}\\ \mathbf{b}_{-i}\sim\mathcal{P}}}\left[u_i^{v_i}(\mathbf{b}_i,\mathbf{b}_{-i})\right] \geq \mathop{\mathbb{E}}_{\substack{\mathbf{b}_i\sim\tilde{\mathcal{D}}\\ \mathbf{b}_{-i}\sim\mathcal{P}}}\left[v_i\Big(x_i(\mathbf{b}_i,\mathbf{b}_{-i})\Big) - \sum_{j\leq x}b_i(j)\right]\\ &= \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\gamma)\Big) - \sum_{j\in\bar{T}_\beta}\beta_j\right] = \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\gamma)\Big)\right] - \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[\sum_{j\in\bar{T}_\beta}\beta_j\right]\end{aligned} \tag{12}$$

By (12), (11), (10) and (9) we derive:

$$\mathop{\mathbb{E}}_{\substack{\mathbf{b}_i\sim\bar{\mathcal{D}}\\ \mathbf{b}_{-i}\sim\mathcal{P}}}\left[u_i^{v_i}(\mathbf{b}_i,\mathbf{b}_{-i})\right] \geq \mathop{\mathbb{E}}_{\substack{\beta\sim\mathcal{D}\\ \gamma\sim\mathcal{D}}}\left[v_i\Big(x_i(\tilde{\beta},\tilde{\gamma})\Big)\right] - \mathop{\mathbb{E}}_{\beta\sim\mathcal{D}}\left[\sum_{1\leq j\leq x}\beta_j\right] \tag{13}$$

The lower bounding by $v_i(x)/2$ of the first term of the right-hand side of this expression is done as previously, for the Discriminatory Auction, in derivations (5) through (6) and the result follows. □

Proof of Theorem 4

Take $n = k$ bidders with the following valuation functions: for any $x \geq 1$, $v_2(x) = 1/k$, $v_i(x) = \epsilon$ for $i \geq 3$, and $v_1(x) = 1$ for $1 \leq x \leq k-1$ and $v_1(k) = 2$. The socially optimum assignment grants k units to bidder 1 and has value 2. Consider the equilibrium

$\mathbf{b}_1 = (1, 0, \ldots, 0)$, $\mathbf{b}_2 = (\frac{1}{k}, 0, \ldots, 0)$ and $\mathbf{b}_i = (\epsilon, 0, \ldots, 0)$ for $i \geq 3$. Every bidder obtains 1 unit under $\mathbf{b}$ and the uniform price is 0. This is a pure Nash equilibrium because bidders $i \geq 2$ do not have any no-overbidding deviation to obtain additional units and no bidder has an incentive to drop their won unit. Bidder 1 may deviate to win all units, e.g., by bidding $\mathbf{b}'_1 = (\frac{1}{k-1}, \ldots, \frac{1}{k-1}, \frac{1}{k-1})$, but he will pay a uniform price $\frac{1}{k}$ for each of the k units, thus not improving upon his utility of 1 under $\mathbf{b}$. The social welfare of $\mathbf{b}$ is $1 + \frac{1}{k} + (k-2) \cdot \epsilon$ which tends to $\frac{k+1}{k}$ as $\epsilon \to 0$. □

Proof of Theorem 5

We first prove the lower bound for the Discriminatory Auction. Fix $k \in \mathbb{N}$ and $\mu \geq 0$ arbitrarily. We construct an instance of the Discriminatory Auction with 2 bidders, submodular valuation functions $\mathbf{v}$ and bidding vectors $\mathbf{b}$, such that for every possible deviation $\mathbf{b}'_i$ of bidder $i = 1, 2$ we have

$$\sum_{i=1}^{2} u_i^{\mathbf{v_i}}(\mathbf{b}'_i, \mathbf{b}_{-i}) \leq \mu \left(1 - \frac{1}{e^{1/\mu}} + \frac{1}{k}\left(1 - \frac{1}{e}\right)\right) SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{j=1}^{k} \beta_j(\mathbf{b}).$$

By taking k to infinity, we see that for any fixed value of μ, the best Price of Anarchy that we can obtain using Theorem 3 is $\max\{1, \mu\}/(\mu(1 - e^{-1/\mu}))$. The latter expression is minimized by taking $\mu = 1$, and from this the claim follows.

The construction of our instance is as follows: Let the valuation functions be defined as $v_1(j) = j$ and $v_2(j) = 0$ for every $j \in [k]$. Then $x_1^{\mathbf{v}} = k$ and $x_2^{\mathbf{v}} = 0$ and the optimal social welfare is $SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) = k$. Define the bid vector $\mathbf{b}_1$ of bidder 1 to be the zero vector and the bid vector $\mathbf{b}_2$ of bidder 2 as

$$b_2(j) = \begin{cases} 1 - \frac{k}{e^{1/\mu}(k-j+1)} & \text{if } 1 \leq j \leq k\left(1 - \frac{1}{e^{1/\mu}}\right) + 1, \\ 0 & \text{if } j > k\left(1 - \frac{1}{e^{1/\mu}}\right) + 1. \end{cases}$$

We assume that the tie-breaking rule of the auction always assigns a unit to bidder 1 when there is a tie. Then, if bidder 2 bids $\mathbf{b}_2$, there is a j between 1 and $k\left(1 - \frac{1}{e^{1/\mu}}\right) + 1$ such that bidder 1 maximizes his utility when he sets all his bids equal to $b_2(j)$. Let $\mathbf{b}'_1 = b_2(j)\mathbf{1}$ for some j in this range. We have

$$\begin{aligned} u_1^{\mathbf{v}_1}(\mathbf{b}'_1, \mathbf{b}_{-2}) &= v_1(k - j + 1) - (k - j + 1)b_2(j) \\ &= \frac{k}{e^{1/\mu}} + \mu \sum_{\ell=1}^{k} b_2(j) - \mu \sum_{\ell=1}^{k} \beta_j(\mathbf{b}) \\ &\leq \frac{k}{e^{1/\mu}} + \mu \int_1^{k(1-e^{-1/\mu})+1} \left(1 - \frac{k}{e^{1/\mu}}(k - t + 1)\right) dt + \\ &\quad \mu \sum_{\ell=1}^{\lceil k(1-\frac{1}{e^{1/\mu}})\rceil} (b_2(\ell) - b_2(\ell + 1)) - \mu \sum_{\ell=1}^{k} \beta_j(\mathbf{b}) \\ &= k\mu\left(1 - \frac{1}{e^{1/\mu}}\right) + b_2(1) - \mu \sum_{\ell=1}^{k} \beta_j(\mathbf{b}) \end{aligned}$$

$$= \mu \left(1 - \frac{1}{e^{1/\mu}} + \frac{1}{k}\left(1 - \frac{1}{e}\right)\right) SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{\ell=1}^{k} \beta_j(\mathbf{b}).$$

For bidder 2, $u_2^{\mathbf{v}_2}(\mathbf{b}'_2, \mathbf{b}_{-1}) \leq 0$ for every bid vector $\mathbf{b}'_2$. This establishes our claim for the Discriminatory Auction.

For the Uniform Price Auction, the construction is more straightforward: we provide an instance of one item and two bidders with submodular valuation functions $\mathbf{v}$ and bidding vectors $\mathbf{b}$ such that for every possible deviation $\mathbf{b}'_i$ of bidder $i = 1, 2$ we have

$$\sum_{i=1}^{2} u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i}) < \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \beta_1(\mathbf{b}). \tag{14}$$

for all $\mu \geq 0$, $\lambda > (1+\mu)/2$. This implies that for any fixed choice of $\mu \geq 0$, the best Price of Anarchy that we can obtain using Theorem 3 is 2.

The instance is as follows. We assume without loss of generality that ties are broken in favor of the second bidder. Suppose $v_1 = 1$ and $v_2 = 1/2$. Let $b_1 = b_2 = v_2 = 1/2$. Now, under $\mathbf{b}$ the second bidder wins the item at a price equal to $\beta_1(\mathbf{b}) = 1/2$. In the optimal allocation, the first bidder wins the item, so $SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) = 1$. Consider an arbitrary deviation b'_1 of bidder 1. If $b'_1 > 1/2$ then $u_1^{v_1}(b'_1, b_2) = 1/2$; otherwise, $u_1^{v_1}(b'_1, b_2) = 0$. Consider an arbitrary deviation b'_2 of bidder 2. We have $u_2^{v_2}(b'_2, b_1) = 0$ because bidder 2 loses if he bids below 1/2 (and by assumption, he does not overbid). Thus, for $\lambda > (1+\mu)/2$ we obtain

$$\sum_{i=1}^{2} u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i}) \leq \frac{1}{2} < \lambda - \mu\frac{1}{2} = \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \beta_1(\mathbf{b}).$$

□

In order to provide some intuition for the above proof for the case of the Discriminatory Auction, let us additionally briefly explain how the instance in Theorem 5 is obtained: Our aim is to construct a two-player instance with bid vector $\mathbf{b}$ such that

$$\sup_{\mathbf{b}'} u_1^{\mathbf{v}_1}(\mathbf{b}'_1, \mathbf{b}_{-1}) + \sum_{j=1}^{x_1^{\mathbf{v}}} \beta_j(\mathbf{b})$$

is as small as possible. For simplicity we restrict ourselves to constant valuation vectors, i.e., there is a constant c such that $\mathbf{v}_1(j) = c$ for all j, and we assume that $x_1^{\mathbf{v}} = k$. Then the summation $\sum_{j=1}^{x_1^{\mathbf{v}}} \beta_j(\mathbf{b})$ in the above equation can be regarded as a Riemann sum of a non-decreasing non-negative function f that nowhere exceeds c. We can assume without loss of generality that the non-zero bids in $\mathbf{b}'$ are equal (as the supremum in the above equation is certainly attained for such a bid vector) and let $n(\mathbf{b}')$ denote the number of non-zero bids in $\mathbf{b}'$. The term $u_1^{\mathbf{v}_1}(\mathbf{b}'_1, \mathbf{b}_{-1})$ can in turn be interpreted as the surface of a rectangle with dimensions $c - f(n(b'))$ and $n(b')$.

The problem of constructing our instance therefore roughly reduces to the geometric problem of finding the right non-negative non-decreasing curve f such that the surface under the curve, plus the maximum surface of a rectangle with dimensions $c - f(n)$ and n for $n \in [0, k]$, is minimized. The proof of Theorem 5, gives essentially a (discretized) description of this curve f, and shows subsequently that with this choice of f the equation above is upper bounded by $(1 - \frac{1}{e})SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}})$ if we take an arbitrarily fine Riemann sum of f (i.e., the Riemann integral), which can be realized by taking k to infinity.

Appendix C: A Lower Bound for the Discriminatory Auction

In this Section, we exhibit that the Bayesian Price of Anarchy is > 1 for the Discriminatory Auction, when we have a discretized strategy space. Consider an instance with 2 bidders and 1 unit. Suppose that there is no uncertainty for the first bidder and his valuation is $v_1 = 1$. For the second bidder, the distribution on his two possible valuation profiles is:

$$v_2 = \begin{cases} v_{21} = 0.667 & \text{with probability } \alpha \\ v_{22} = 0.333 & \text{with probability } 1 - \alpha \end{cases}$$

We use as a tie-breaking rule that ties are always resolved in favor of player 1. We assume that the bidding space is discretized so that all bids should be multiples of ϵ. We can make ϵ as small as we want but it should be some fixed number. For this example, we will assume that $\epsilon = 0.001$ and that $\alpha = 0.0014$.

We claim that the following profile is a pure Bayes-Nash equilibrium: Player 1 plays the profile $b_1(v_1) = 0.333$ and player 2 plays $b_2(v_{21}) = 0.333 + \epsilon$, and $b_2(v_{22}) = 0.333$.

The idea behind the example is that because α is very small, player 1 realizes that most of the times, he can win the item by bidding no more than 1/3. But then, with some very small probability, player 2 wins the item and we end up with an inefficient assignment. To see that this is a Bayes-Nash equilibrium, we compute the expected utilities of the two players for each valuation profile:

$$\mathbb{E}[u_1^{v_1}(\mathbf{b})] = (1 - \alpha)(1 - 0.333) = 0.667(1 - \alpha)$$

$$\mathbb{E}[u_2^{v_{21}}(\mathbf{b})] = 0.667 - (0.333 + \epsilon) = 0.334 - \epsilon, \quad \mathbb{E}[u_2^{v_{22}}(\mathbf{b})] = 0$$

It is quite easy to see that player 2 does not have an incentive to deviate regardless of his actual valuation. We now check that player 1 also does not have an incentive to deviate. Player 1 will never have an incentive to bid higher than $0.333 + \epsilon$, given the bid of player 2 and the tie-breaking we use. Hence there are only 2 possible deviations to consider: if he decides to bid lower than his current bid, he does not win the item and drops to a zero utility. The only interesting case is the second one, where he decides to bid $c = 0.333 + \epsilon$.

In that case he always wins the item but pays ϵ more. His expected utility is

$$\mathbb{E}[u_1^{v_1}(c, \mathbf{b}_{-1})] = 1 - 0.333 - \epsilon = 0.667 - \epsilon$$

It can be easily verified now that as long as $\alpha \leq 1.499 \cdot \epsilon$, it holds that $\mathbb{E}[u_1^{v_1}(c, \mathbf{b}_{-1})] \leq \mathbb{E}[u_1^{v_1}(\mathbf{b})]$.

To see now what is the price of Anarchy, note that $\mathbb{E}_{\mathbf{v}}[OPT] = 1$, whereas

$$E_{\mathbf{v}}[SW(\mathbf{v}, \mathbf{b}^{\mathbf{v}})] = \alpha \cdot 0.667 + (1 - \alpha) \cdot 1 = 1 - 0.333\alpha$$

For the values we selected for α and ϵ we see that the Bayesian Price of Anarchy is at least 1.0004.

Note: It does not make a big difference in this example if we replace 0.667 and 0.333 with other numbers. The bound we get is still very close to 1.

APPENDIX D: Omitted Proofs from Section 4

Proof of Theorem 7

Note that for the Discriminatory Auction we have

$$\sum_{i\in[n]} P_i(\mathbf{b}) = \sum_{i\in[n]} \sum_{j=1}^{x_i(\mathbf{b})} b_i(j) = \sum_{j=1}^{k} \beta_j(\mathbf{b}).$$

Note that Lemma 1 holds for every bidder $i \in [n]$ with λ and μ as stated in Theorem 7 (see also the proof of Theorem 1). By invoking Lemma 1 and summing inequality (2) over all bidders, we obtain

$$\sum_{i\in[n]} \mathbb{E}[u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i})] \geq \lambda \sum_{i\in[n]} v_i(x_i^{\mathbf{v}}) - \mu \sum_{i\in[n]} \sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\mathbf{b}_{-i}) \geq \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{j=1}^{k} \beta_j(\mathbf{b}),$$

where the last inequality holds because for every bidder i, $\beta_j(\mathbf{b}_{-i}) \leq \beta_j(\mathbf{b})$ for every $j = 1, \dots, k$, and $\sum_{i\in[n]} x_i(\mathbf{b}) = k$ and $\beta_j(\mathbf{b})$ is non-decreasing in j. $\square$

Proof of Theorem 9

The following transformation will be helpful in the poof of Theorem 9. Let $\mathbf{b}$ be an arbitrary bidding profile. We derive a uniform bidding profile $\bar{\mathbf{b}}$ from $\mathbf{b}$ as follows: Let $c_i = b_i(x_i(\mathbf{b}))$ be the last winning bid of bidder $i \in [n]$; for ease of notation, we adopt the convention that $b_i(0) = 0$. Define $\bar{\mathbf{b}}_i$ as the vector that is c_i on the first $x_i(\mathbf{b})$ entries and zero everywhere else. Clearly, $\bar{\mathbf{b}}$ is a uniform bidding profile.

Lemma 6 *Let $\bar{\mathbf{b}}$ be the uniform bidding profile derived from $\mathbf{b}$. Then for the Uniform Price Auction, the following holds for every bidder $i \in [n]$:*

(i) $x_i(\bar{\mathbf{b}}) = x_i(\mathbf{b})$;

(ii) $B_i(\bar{\mathbf{b}}_i, x_i(\bar{\mathbf{b}})) = B_i(\mathbf{b}_i, x_i(\mathbf{b}))$.

Proof. Let β_1 and β_0 refer to the smallest winning bid and largest losing bid under bidding profile $\mathbf{b}$, respectively. Note that every winning bid under $\bar{\mathbf{b}}$ is at least β_1 and can only increase under $\mathbf{b}$. Also, every losing bid under $\bar{\mathbf{b}}$ is at most β_0 under $\mathbf{b}$. We conclude that a bid is winning under $\bar{\mathbf{b}}$ if and only if it is winning under $\mathbf{b}$, which proves (i).

Recall that $B_i(\bar{\mathbf{b}}_i, x_i(\bar{\mathbf{b}})) = x_i(\bar{\mathbf{b}})\bar{b}_i(x_i(\bar{\mathbf{b}}))$. Now, using (i) and the definition of $\bar{\mathbf{b}}_i$, we obtain: $B_i(\bar{\mathbf{b}}_i, x_i(\bar{\mathbf{b}})) = x_i(\bar{\mathbf{b}})\bar{b}_i(x_i(\bar{\mathbf{b}})) = x_i(\mathbf{b})\bar{b}_i(x_i(\mathbf{b})) = x_i(\mathbf{b})b_i(x_i(\mathbf{b})) = B_i(\mathbf{b}_i, x_i(\mathbf{b}))$, which shows (ii). $\square$

We can now prove Theorem 9:

Proof. [of Theorem 9] We first prove weak smoothness for the uniform bidding format and then extend this result to the standard bidding format via a coupling argument.

It is not hard to verify that for the Uniform Price Auction we have $B_i(\mathbf{b}_i, x) = xb_i(x)$. As before, exploiting Lemma 1 and summing inequality (2) over all bidders we obtain

$$\sum_{i\in[n]} \mathbb{E}[u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i})] \geq \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{j=1}^{k} \beta_j(\mathbf{b}).$$

If $\mathbf{b}$ is a uniform bidding profile then the claim follows because

$$\sum_{j=1}^{k} \beta_j(\mathbf{b}) = \sum_{i\in[n]} x_i(\mathbf{b}) b_i(x_i(\mathbf{b})) = \sum_{i\in[n]} B_i(\mathbf{b}_i, x_i(\mathbf{b})). \tag{15}$$

Note that for the standard bidding format, the first equality would be false because we can only infer that $\sum_j \beta_j(\mathbf{b}) \geq \sum_i x_i(\mathbf{b}) b_i(x_i(\mathbf{b}))$. However, the following work-around establishes weak smoothness for the Uniform Price Auction and the standard bidding format.

Note that in general $P_i(\bar{\mathbf{b}}) \neq P_i(\mathbf{b})$ because all losing bids in $\bar{\mathbf{b}}$ are zero. However, the above two properties turn out to be sufficient to prove weak smoothness in the standard bidding format: Let $\bar{\mathbf{b}}$ be the uniform bidding profile that we obtain from $\mathbf{b}$ as described above. Applying Lemma 1 to the uniform bidding profile $\bar{\mathbf{b}}$ and pricing rule $P_i(\bar{\mathbf{b}}) = B_i(\bar{\mathbf{b}}, x_i(\bar{\mathbf{b}}))$ (which is discriminatory price dominated), we conclude that for every bidder $i \in [n]$ there exists a random uniform bidding profile $\mathbf{b}'_i$ such that

$$\mathbb{E}[v_i(x_i(\mathbf{b}'_i, \bar{\mathbf{b}}_{-i})) - B_i(\mathbf{b}'_i, x_i(\mathbf{b}'_i, \bar{\mathbf{b}}_{-i}))] \geq \lambda v_i(x_i^{\mathbf{v}}) - \mu \sum_{j=1}^{x_i^{\mathbf{v}}} \beta_j(\bar{\mathbf{b}}_{-i}). \tag{16}$$

Note that by Lemma 6

$$\begin{aligned} u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i}) &= v_i(x_i(\mathbf{b}'_i, \mathbf{b}_{-i})) - P_i(\mathbf{b}'_i, \mathbf{b}_{-i}) &\quad (17)\\ &\geq v_i(x_i(\mathbf{b}'_i, \mathbf{b}_{-i})) - B_i(\mathbf{b}'_i, x_i(\mathbf{b}'_i, \mathbf{b}_{-i})) \\ &= v_i(x_i(\mathbf{b}'_i, \bar{\mathbf{b}}_{-i})) - B_i(\mathbf{b}'_i, x_i(\mathbf{b}'_i, \bar{\mathbf{b}}_{-i})). &\quad (18) \end{aligned}$$

By summing inequality (16) over all bidders and exploiting (18), we thus obtain

$$\begin{aligned} \sum_{i\in[n]} \mathbb{E}[u_i^{\mathbf{v}_i}(\mathbf{b}'_i, \mathbf{b}_{-i})] \geq \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{j=1}^{k} \beta_j(\bar{\mathbf{b}}) &= \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{i\in[n]} B_i(\bar{\mathbf{b}}_i, x_i(\bar{\mathbf{b}})) \\ &= \lambda SW(\mathbf{v}, \mathbf{x}^{\mathbf{v}}) - \mu \sum_{i\in[n]} B_i(\mathbf{b}_i, x_i(\mathbf{b})) \end{aligned}$$

where the first equality follows from (15) because $\bar{\mathbf{b}}$ is a uniform bidding profile and the second equality holds because of Lemma 6. $\square$